JEE Main - 2023

Mathematics

Answers

Section A

1. Correct Answer: C

Since the divisor is perfect square each prime factor must occur even number of times.

2 can be taken in 2 ways (2^0 or 2^2)

3 can be taken in 3 ways (3^0 or 3^2 or 3^4)

Similarly 5 can be taken in 4 ways (5^0 or 5^2 or 5^4 or 5^6)

and 7 can be taken in 4 ways (7^0 or 7^2 or 7^4 or 7^6)

hence total divisor which are perfect squares

$$=2\cdot3\cdot4\cdot4=96$$

2. Correct Answer: D

$$g(x)=\begin{cases} 0 & ,-2\le x<0 \\ \left|x^2-1\right|+x^2-1, & 0\le x<2 \end{cases}$$

$\therefore$ Between 0 & 2 $g(x)$ is non- differentiable at $x=1$ as $\left|x^2-1\right|$ is ND at $x=1$ &
$ND\pm D=ND$

3. Correct Answer: B

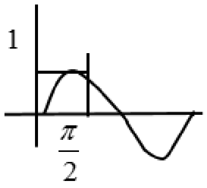
To reach point B from A, the insert has to crawl either right or downward More specifically , insert has to crawl 5 units rightward & 3 units downward.

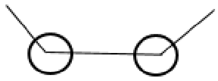
We have $5(\rightarrow)$ & $3(\downarrow)$

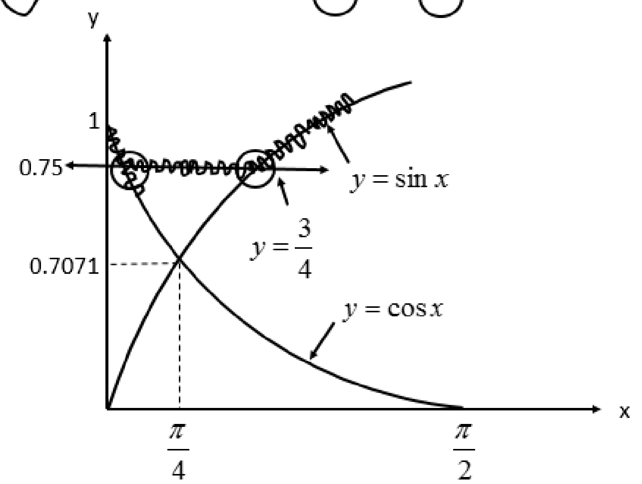
$\frac{8!}{5!3!}\rightarrow$ No of possible paths.

$$= \frac{8 \times 7 \times \cancel{6} \times \cancel{5!}}{\cancel{5!} \times \cancel{3 \times 2}} = 56$$

4. Correct Answer: B







$$\frac{1}{\sqrt{2}} \simeq \frac{1}{1.414} \approx 0.707$$

$$\sin x = \cos x$$

$$\tan = x = 1$$

$$y = \frac{3}{4}$$

$$\boxed{y = 0.75}$$

5. Correct Answer: D

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$$

(Apply $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$)

$$=\begin{vmatrix}1 & 0 & 0\\ 1 & x & 0\\ 1 & 0 & y\end{vmatrix}=1(xy-0)=xy$$

Hence D is divisiable by both x and y

6. Correct Answer: C

$x^2+ax+1=0$

Let roots be α and β, then $\alpha+\beta=-a$ and $\alpha\beta=1$

$|\alpha-\beta|=\sqrt{(\alpha+\beta)^2-4\alpha\beta}, |\alpha-\beta|=\sqrt{a^2-4}$

Since, $|\alpha-\beta|<\sqrt{5}\Rightarrow\sqrt{a^2-4}<\sqrt{5}$

$\Rightarrow a^2-4<5\Rightarrow a^2<9\Rightarrow -3<a<3$

7. Correct Answer: C

$\sum_{r=0}^{n}(r+1)^n C_r \sum_{r=0}^{n} r\cdot{}^n C_r+\sum_{r=0}^{n}{}^n C_r$

$=\sum_{r=0}^{n} r\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1}+\sum_{r=0}^{n}{}^n C_r$

$=n\cdot 2^{n-1}+2^n=2^{n-1}(n+2)$

Thus Statement -1 is true.

Again $\sum_{r=0}^{n}(r+1)^n C_r x^r=\sum_{r=0}^{n} r\cdot{}^n C_r x^r+\sum{}^n C_r x^r$

$=n\sum_{r=0}^{n}{}^{n-1}C_{r-1}x^r+\sum_{r=0}^{n}{}^n C_r x^r$

$=nx(1+x)^{n-1}+(1+x)^n$

Substitute $x=1$ in the above identity to get $\sum(r+1)^n C_r=n\cdot 2^{n-1}+2^n$

Statement -2 is also true & explains Statement -1 also.

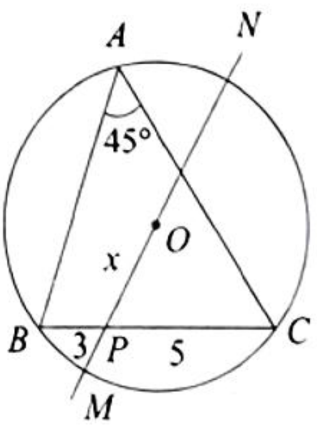
8. Correct Answer: D

$$\left[\sec^{-1} t\right]_{\sqrt{2}}^{x} = \frac{\pi}{2}$$

$$\sec^{-1} x - \sec^{-1}\sqrt{2} = \frac{\pi}{2} \Rightarrow \sec^{-1} x = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

$x = -\sqrt{2}$. There is no correct option

9. Correct Answer: B



Let $OP = x$

Also, let circumradius $= R$

$$\therefore \frac{BC}{\sin 45^\circ} = 2R \Rightarrow R = 4\sqrt{2}$$

Now, $PB \times PC = PM \times PN$

$$\Rightarrow 15 = (R - x)(R + x)$$

$$\Rightarrow x = \sqrt{17}$$

Hence, the correct answer is (B)

10. Correct Answer: A

n^{th} term of the series is $20+(n-1)\left(-\frac{2}{3}\right)$

For the sum to be maximum

n^{th} term ≥ 0

$\Rightarrow 20+(n-1)\left(-\frac{2}{3}\right)\geq 0$

$\Rightarrow n\leq 31$

Thus, the sum of 31 terms is maximum and is

$\frac{31}{2}\left[40+30\times\left(-\frac{2}{3}\right)\right]=310$

11. Correct Answer: C

$\log_a b=\frac{\log b}{\log a}$

Let $b=2^m$ and $a=2^n$ where m and n denotes the exponents on the base 2 in the given set. Then $\log_a b=\frac{m}{n}$

Therefore, $\log_a b$ is an integer only if n divides m.

Now, total no. of ways m and n can be chosen $=25\times 24=600$

For favourable cases,

Let	$n=1$	So m can take values $1,2,3,4,5,6,.....,24$	$=24$
If	$n=2$	$m=4,6,8,10,12,14,16,18,20,22,24$	$=11$
	$n=3$	$m=6,9,12,15,18,21,24$	$=7$
	$n=4$	$m=8,12,16,20,24$	$=5$
	$n=5$	$m=10,15,20,25$	$=4$
	$n=6$	$m=12,18,24$	$=3$
	$n=7$	$m=14,21$	$=2$
	$n=8$	$m=16,24$	$=2$
	$n=9,10,11,12$	$m=1$ for each	$=4$
			Total $=62$

Hence, required probability $=\frac{62}{600}=\frac{31}{300}$

12. Correct Answer: B

The given limit

$$L=\lim_{n\to\infty}\sum_{r=1}^{n} n.\frac{1}{(n+r)(n+2r)}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right)\left(n+\frac{2r}{n}\right)}$$

$$=\int_0^1\frac{dx}{(1+x)(1+2x)}$$

$$=\int_0^1\left(\frac{-1}{1+x}+\frac{2}{1+2x}\right)dx$$

$$=\left[-\log(1+x)+\log(1+2x)\right]_0^1$$

$$=\left[(-\log 2+\log 3)-(-\log 1+\log 1)\right]$$

$$=\log\left(\frac{3}{2}\right)$$

13. Correct Answer: B

Let $f(x)=\frac{1-x+x^2}{1+x+x^2}$

$$\therefore f'(x)=\frac{\left(1+x+x^2\right)(-1+2x)-\left(1-x+x^2\right)(1+2x)}{\left(1+x+x^2\right)^2}$$

$$=\frac{2x^2-2}{\left(1+x+x^2\right)^2}=\frac{2\left(x^2-1\right)}{\left(1+x+x^2\right)^2}$$

$$=\frac{2(x-1)(x+1)}{\left(1+x+x^2\right)^2}$$

$f'(x) > 0$ for $x < -1$ or $x > 1$

$f'(x) < 0$ for $-1 < x < 1$

Thus $f'(x)$ changes sign from $-ve$ to $+ve$ at $x = 1$

Also, $f'(x) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

So, $x = 1$ is point of minima

Minimum value is given by $f(1) = \dfrac{1-1+1}{1+1+1} = \dfrac{1}{3}$

14. Correct Answer: C

As we know

$\cos^{-1} A - \cos^{-1} B$

$= \cos^{-1}\left(AB + \sqrt{1-A^2}.\sqrt{1-B^2}\right)$

Given, $\cos^{-1} x - \cos^{-1}\dfrac{y}{2} = \alpha$

$$\Rightarrow \cos^{-1}\left(x.\frac{y}{2} + \sqrt{1-x^2}.\sqrt{1-\frac{y^2}{4}}\right) = \alpha$$

$$\Rightarrow \frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = \cos x$$

$$\Rightarrow \left(\cos x - \frac{xy}{2}\right)^2 = \left(1-x^2\right)\left(1-\frac{y^2}{4}\right)$$

$$\Rightarrow \cos^2 + \frac{x^2y^2}{4} - 2.\cos x.\frac{xy}{2}$$

$$= 1 - x^2 - \frac{y^2}{4} + \frac{x^2y^2}{4}$$

$$\Rightarrow x^2 + \frac{y^2}{4} - xy\cos x = 1 - \cos^2 x$$

$$\Rightarrow x^2+\frac{y^2}{4}-xy\cos x=\sin^2 x$$

$$\Rightarrow 4x^2+y^2-4xy\cos x=4\sin^2 x$$

15. Correct Answer: D

$$8\cos x\left[\frac{1}{2}\left(\cos\frac{\pi}{3}+\cos 2x\right)-\frac{1}{2}\right]=1$$

$$=8\cos x\left[\frac{1}{2}\left(\frac{1}{2}+1-2\sin^2 x\right)-\frac{1}{2}\right]=1$$

$$=8\cos x\left[\frac{1}{2}\left(\frac{3}{2}-2\sin^2 x\right)-\frac{1}{2}\right]=1$$

$$=8\cos x\left[\frac{3}{4}-\sin^2 x-\frac{1}{2}\right]=1$$

$$=8\cos x\left(\frac{1}{4}-\sin^2 x\right)=1$$

$$=8\cos x\left(\frac{1}{4}-1+\cos^2 x\right)=1$$

$$=8\cos x\left(\cos^2 x-\frac{3}{4}\right)=1$$

$$=8\left[\frac{4\cos^3 x-3\cos x}{4}\right]=1$$

$$=2\cos 3x=1\Rightarrow \cos 3x=\frac{1}{2}$$

$$\therefore 3x=2n\pi\pm\frac{\pi}{3}$$

$$\therefore x=\frac{2n\pi}{3}\pm\frac{\pi}{9}$$

In $[0,2\pi]$

$x=\frac{\pi}{9},\frac{7\pi}{9},\frac{13\pi}{9},\frac{5\pi}{9},\frac{11\pi}{9},\frac{17\pi}{9}$

$\text{sum}=(1+5+7+11+13+17)\frac{\pi}{9}$

$=6\pi=k\pi$

$\therefore \boxed{k=6}$

16. Correct Answer: B

$I=\int\frac{(\cos x-\sin x)}{\sqrt{8-\sin 2x}}dx$

Consider,

$1+\sin 2x$

$=\sin^2 x+2\sin x\cdot\cos x+\cos^2 x$

$1+\sin 2x=(\sin x+\cos x)^2 \Rightarrow \sin 2x=(\sin x+\cos x)^2-1$

Now, put $\sin x+\cos x=t$

$(\cos x-\sin x)dx=dt$

$\therefore I=\int\frac{1 \quad dt}{\sqrt{8-(t^2-1)}}$

$I=\int\frac{1}{\sqrt{9-t^2}}dt=\sin^{-1}\left(\frac{t}{3}\right)+c$

$I=\sin^{-1}\left(\frac{\sin x+\cos x}{3}\right)+c$

Comparing with a $\sin^{-1}\left(\frac{\sin x+\cos x}{2b}\right)+c$, we get

$a=1, b=\frac{3}{2}$

17. Correct Answer: C

Since the given circle passes through $(6,1)$ it will satisfy the equation of circle.

$36+1-12g+6-19c=0$

$12g+19c=43$ (1)

Centre of given circle is $(g,-3)$. It will satisfy the line $x-2cy=8$

$\therefore 9+6c=8c$ (2)

From (1) & (2), we get

$g=2, c=1$

$\therefore$ Equation of circle is $x^2+y^2-4x+6y-19=0$

y-intercept $=2\sqrt{f^2-c}$

$=2\sqrt{(3)^2-(-19)}$

$=2\sqrt{28}$

18. Correct Answer: C

$f(x)=x\cdot e^{x(1-x)}$

$f'(x)=x\left[e^{x(1-x)}(1-2x)\right]+e^{x(1-x)}$

$=e^{x(1-x)}\left[x-2x^2+1\right]$

$=-e^{x(1-x)}\left[2x^2-x-1\right]$

$f'(x)=-e^{x(1-x)}(2x+1)(x-1)$

$\therefore f(x)$ is increasing in $\left(\frac{-1}{2},1\right)$ and

decreasing in $\left(-\infty,\frac{-1}{2}\right)\cup(1,\infty)$

19. Correct Answer: B

Integrating Factor, $I.F = e^{\int \frac{2x^2+11x+13}{x^3+6x^2+11x+6}dx}$

Now, $\frac{2x^2+11x+13}{x^3+6x^2+11x+6} = \frac{2x^2+11x+13}{(x+1)(x+2)(x+3)}$

Let, $\frac{2x^2+11x+13}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

$2x^2+11x+13 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

Put $x=-2$, we get

$-1 = B(-1) \Rightarrow \boxed{B=1}$

Put $x=-3$, we get

$-1 = 2C \Rightarrow \boxed{C=-1}$

Put $x=-1$, we get

$4 = 2A \Rightarrow \boxed{A=2}$

Integrating factor, If $= e^{\int\left(\frac{2}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}\right)}$

If $= e^{2\cdot\log\|x+1\|+\log|x+2|-\log|x+3|}$

$= e^{\log_e\left|\frac{(x+1)^2\cdot(x+2)}{(x+3)}\right|}$

If $= \frac{(x+1)^2(x+2)}{(x+3)}$

Solution of Differential Equation,

$$y\cdot\frac{(x+1)^2(x+2)}{(x+3)} = \int\frac{(x+3)}{(x+1)}\times\frac{(x+1)^2(x+2)}{(x+3)}dx$$

$$y\cdot\frac{(x+1)^2(x+2)}{(x+3)} = \int(x+1)(x+2)\,dx$$

$$\frac{y(x+1)^2(x+2)}{(x+3)} = \frac{x^3}{3} + \frac{3x^2}{2} + 2x + c$$

Curve passes through $(0,1)$

$$\frac{1\times1\times2}{3} = O + C$$

$$\therefore C = \frac{2}{3}$$

$$y(2) = \frac{\frac{8}{3}+6+4+\frac{2}{3}}{\frac{9\times4}{5}} = \frac{\frac{10}{3}+10}{\frac{36}{5}} = \frac{40}{3}\times\frac{5}{36}$$

$$= \frac{50}{27}$$

20. Correct Answer: C

Equation of tangent at vertex id $x + y - a = 0$

Focus, $F = (a,a)$

Length of latus Return $= 4x$ (Perpendicular distance of tangent from focus F)

$$= 4\times\left|\frac{a+a-a}{\sqrt{1^2+1^2}}\right|$$

$$24 = 4\left|\frac{a}{\sqrt{2}}\right|$$

$$\therefore |a| = 6\sqrt{2}$$

Section B

21. Correct Answer: 4

Let I be the set of Indian people and F the set of foreigners W stands for Women, M for Men and D for Doctors.

$I = W \cup M$

$\therefore n(I) = n(W \cup M)$

$= n(W) + n(M) - n(W \cap M)$

$= 29 + 23 - n(\phi)$ ($W \cap M = \phi$ as a man cannot be a woman.)

$= 52 - 0 = 52$

Foreign delegates = Total-Indian $= 100 - 52 = 48$

Again we are given $n(M \cup D) = 24$

or $n(M) + n(D) - n(M \cap D) = 24$

$23 + 5 - n(M \cap D) = 24$

$\therefore n(M \cap D) = 4$

Hence amongst the set of men there are four doctors. Since the total number of doctors (Indian only) is 5 out of which we have shown 4 are men and hence we conclude there is only one woman (Indian) doctor. There are no foreign doctors.

22. Correct Answer: 1

$l = \lim_{x \to 0} x^x$

$\log L = \lim_{x \to 0} \log(x^x)$

$\log L = \lim_{x \to 0} x . \log x$

$\log L = \lim_{x \to 0} \frac{\log x}{\frac{1}{x}}$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = -x$$

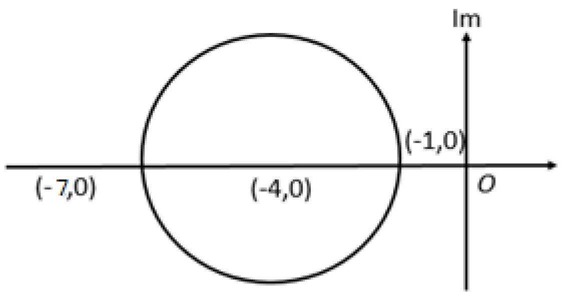
$\log L = 0$

$\therefore L = e^{\circ} = 1$

23. Correct Answer: 6

z lies on or inside the circle with centre $(-4, 0)$ and radius 3 units.

Hence maximum distance of z from $(-7, 0)$ id 6 units.



24. Correct Answer: 14

Let the observation be $x_1, x_2, x_3, ..., x_{10}$

Mean $= \frac{x_1 + x_2 + ...+ x_{10}}{10} = 20$

$\therefore x_1 + x_2 + ...+ x_{10} = 200$ (1)

Variance $= \frac{x_1^2 + x_2^2 + ...+ x_{10}^2}{10} - (20)^2 = 15$

$\therefore \; = x_1^2 + x_2^2 + ...+ x_{10}^2 = 4150$

Now,

Corrected Sum $= 20D - 25 + 15 = 190$

Corrected Mean $=\dfrac{190}{10}=19$

$\therefore$ Corrected Variance $=\dfrac{x_1^2+.....+x_{10}^2-25^2+15^2}{10}-(19)^2$

$=\dfrac{4150-625-225}{10}-361$

$=14$

$\therefore$ Corrected standard deviation $=\sqrt{14}=3.74$

25. Correct Answer: 2500

Since the man is walking at constant speed, and reaches point Q from P in a straight line in the minimum time

$\therefore PR+Q'R$ has to be a straight line

$Q'(0,-2)$ is the image of $Q(0,2)$

Equation of line PQ':

$y+2=\left(\dfrac{4+2}{-3}\right)(x-0)$

$y+2=-2x$

$\therefore$ To find co-ordinates of R, put $y=0$

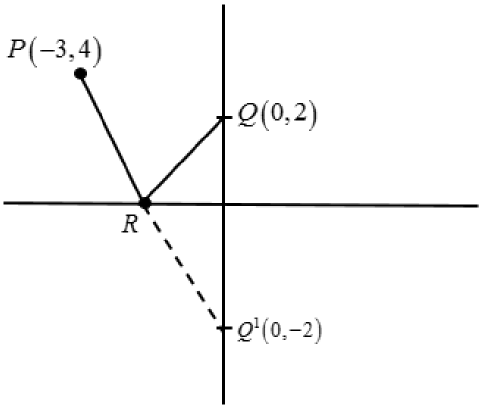
$\therefore x=-1$

$\therefore R\equiv(-1,0)$

$\therefore PR^2=(-3+1)^2+(4-0)^2=20$

$QR^2=(0+1)^2+(2-0)^2=5$

$\therefore 100\left(PR^2+QR^2\right)=100(20+5)=2500$



26. Correct Answer: 58

Equation of Ellipse is:

$$(x+1)^2 + 4(y+1)^2 = \lambda + 5$$

i. e, $\dfrac{(x+1)^2}{(\lambda+5)} + \dfrac{(y+1)^2}{\left(\dfrac{\lambda+5}{4}\right)} = 1$

Length of latus Return $= \dfrac{2\left(\dfrac{\lambda+5}{4}\right)}{\sqrt{\lambda+5}} = \dfrac{7}{2}$

$$\therefore \frac{\sqrt{\lambda+5}}{2} = \frac{7}{2}$$

$$\lambda + 5 = 49$$

$$\lambda = 44$$

Now,

Length of major axis, $l = 2\sqrt{\lambda+5}$

$$= 2\sqrt{44+5} = 2\times 7$$

$$l = 14$$

$$\therefore \lambda + l = 44 + 14 = 58$$

27. Correct Answer: 63

$y^2 = 2x$

$V(0,0)$ and $S = \left(\frac{1}{2}, 0\right)$

Let equation of circle be

$(x-h)^2 + (y-k)^2 = 4$

$\because$ Circle passes through $(0,0)$

$h^2 + k^2 = 4$ (1)

$\because$ Circle passes through $\left(\frac{1}{2}, 0\right)$

$\left(\frac{1}{2} - h\right)^2 + k^2 = 4$

$h^2 + k^2 - h = \frac{15}{4}$ (2)

On solving (1) & (2), we get

$h = \frac{1}{4}, k = \pm\frac{\sqrt{63}}{4}$

But, $\left(\frac{1}{4}, \frac{-\sqrt{63}}{4}\right)$ can't touch given parabola

$\therefore$ Equation of circle is

$\left(x - \frac{1}{4}\right)^2 + \left(k - \frac{\sqrt{63}}{4}\right)^2 = 4$

$\therefore \alpha = 2 + \frac{\sqrt{63}}{4} = \frac{8+\sqrt{63}}{4}$

$\therefore 4\alpha - 8 = \sqrt{63}$

$\therefore (4\alpha - 8)^2 = 63$

28. Correct Answer: 126

$$|adj(adj\ A)| = \left(|A|^2\right)^2 = |A|^4 = \begin{bmatrix} 14 & 28 & -14 \\ -14 & 14 & 28 \\ 28 & -14 & 14 \end{bmatrix}$$

$$\therefore |A|^4 = (14)^3 \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= (14)^3 3(-2(-5)-1(-1))$$

$$|A|^4 = (14)^4$$

$$\therefore |A| = 14$$

$$\therefore |3A| = 3^3 |A| = 9 \times 14 = 126$$

29. Correct Answer: 16

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2|\vec{b}|^2$$

$$|\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + 2|\vec{b}|^2$$

$$2(3) = |\vec{b}|^2$$

$$\therefore |\vec{b}|^2 = 6$$

Now,

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a}.\vec{b})^2$$

$$87 = |\vec{a}|^2 \times 6 - 9$$

$$\therefore |\vec{a}|^2 = \frac{96}{6}$$

$$\boxed{|\vec{a}|^2 = 16}$$

30. Correct Answer: 4

$p^4 + q^4 = 369$

$= \left[(p+q)^2 - 2pq \right]^2 - 2p^2q^2 = 369$

$\left((3)^2 - 2pq \right)^2 - 2(pq)^2 = 369$

$(q - 2pq)^2 - 2(pq)^2 = 369$

$(pq)^2 - 18pq - 144 = 0$

$pq = -6$ or 24

But $pq \neq 24$

$\therefore pq = -6$

$\therefore \left(\frac{1}{p} + \frac{1}{q} \right)^{-2}$

$= \left(\frac{pq}{p+q} \right)^2 = (-2)^2 = 4$