JEE ADVANCED-2012

MATHEMATICS

41. **Sol. (B)**

Given $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$

$\Rightarrow \lim_{x\to\infty}\frac{x^2+x+1-ax^2-ax-bx-b}{(x+1)}=4 \Rightarrow \lim_{x\to\infty}\frac{(1-a)x^2+(1-a-b)x+(1-b)}{(x+1)}=4$

$\Rightarrow 1-a=0$ and $1-a-b=4 \Rightarrow b=-4, a=1$.

42. **Sol. (D)**

$$|Q|=\begin{vmatrix}2^2a_{11} & 2^3a_{12} & 2^4a_{13}\\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23}\\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33}\end{vmatrix} \Rightarrow |Q|=2^2\cdot 2^3\cdot 2^4\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ 2a_{21} & 2a_{22} & 2a_{23}\\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33}\end{vmatrix}$$

$$|Q|=2^9\cdot 2\cdot 2^2\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$\Rightarrow |Q|=2^{12}|P|$

$|Q|=2^{13}$.

43. **Sol. (A)**

Equation of the chord bisected at $P(h,k)$

$hx+ky=h^2+k^2 \quad \ldots(i)$

Let any point on line be $\left(\alpha, \frac{4}{5}\alpha-4\right)$

Equation of the chord of contact is

$\Rightarrow \alpha x+\left(\frac{4}{5}\alpha-4\right)y=9 \quad \ldots(iii)$

Comparing (i) and (ii)

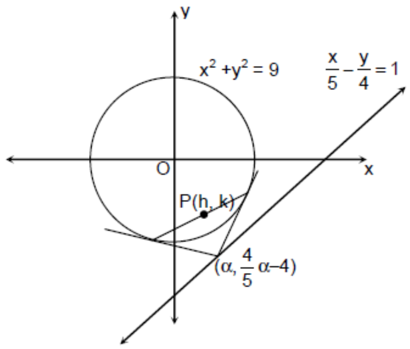
$\frac{h}{\alpha}=\frac{k}{\frac{4}{5}\alpha-4}=\frac{h^2+k^2}{9}$

$\alpha=\frac{20h}{4h-5k}$

Now, $\frac{h(4h-5k)}{20h}=\frac{h^2+k^2}{9}$

$20(h^2+k^2)=9(4h-5k)$

$20(x^2+y^2)-36x+45y=0$



44. **Sol. (B)**

Number of ways

$$= 3^5 - {}^3C_1 2^5 + {}^3C_2 1^5$$

$$= 243 - 96 + 3 = 150.$$

45. **Sol. (C)**

$$I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$$

Let $\sec x + \tan x = t$

$$\Rightarrow \sec x - \tan x = 1/t$$

Now $\left(\sec x \tan x + \sec^{2x}\right) dx = dt$

$$\sec x(\sec x + \tan x) dx = dt$$

$$\sec x\, dx = \frac{dt}{t}, \frac{1}{2}\left(t + \frac{1}{t} = \sec x\right)$$

$$I = \frac{1}{2}\int \frac{\left(t + \frac{1}{t}\right)}{t^{9/2}} \frac{dt}{t}$$

$$= \frac{1}{2}\int \left(t^{-9/2} + t^{-13/2}\right) dt$$

$$= \frac{1}{2}\left[\frac{t^{-9/2+1}}{-\frac{9}{2}+1} + \frac{t^{-13/2+1}}{-\frac{13}{2}+1}\right]$$

$$=\frac{1}{2}\left[\frac{t^{-7/2}}{-\frac{7}{2}}+\frac{t^{-11/2}}{-\frac{11}{2}}\right]$$

$$=-\frac{1}{7}t^{-7/2}-\frac{1}{11}t^{-11/2}$$

$$=-\frac{1}{7}\frac{1}{t^{-7/2}}-\frac{1}{11}\frac{1}{t^{-11/2}}$$

$$=-\frac{1}{t^{-11/2}}\left(\frac{1}{11}+\frac{t^2}{7}\right)=-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+k$$

46. **Sol. (A)**

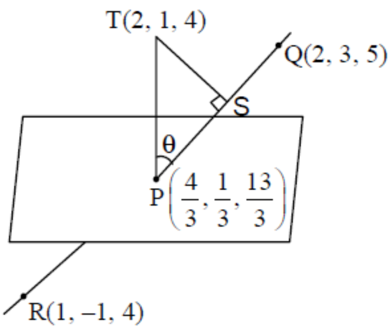
$D.R.$ of QR is $1,4,1$

Coordinate of $P\equiv\left(\frac{4}{3},\frac{1}{3},\frac{13}{3}\right)$

$D.R.$ of PT is $2,2,-1$

Angle between QR and PT is 45° And $PT=1$

$$\Rightarrow PS=TS=\frac{1}{\sqrt{2}}$$



47. **Sol. (B)**

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right|-0}{h}$$

$$=\lim_{h\to 0}h\cos\left(\frac{\pi}{h}\right)=0$$

so, $f(x)$ is differentiable at $x=0$

$$f'(2^+)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2\left|\cos\frac{\pi}{2+h}\right|-0}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$f'(2^+)=\lim_{h\to 0}\frac{(2+h)^2}{h}\sin\left(\frac{\pi}{2}-\frac{\pi}{2+h}\right)$$

$$=\lim_{h\to 0}\frac{(2+h)^2}{h}\sin\left[\frac{\pi\cdot h}{2(2+h)}\right]$$

$$=\lim_{h\to 0}\frac{(2+h)^2}{\frac{\pi h}{2(2+h)}}\sin\frac{\pi h}{2(2+h)}\times\frac{\pi}{2(2+h)}=\pi$$

Again, $f'(2^-)=\lim_{h\to 0}\dfrac{f(2-h)-f(2)}{-h}$

$$= \lim_{h\to 0} \frac{(2+h)^2 \left|\cos\left(\frac{\pi}{2-h}\right)\right|}{-h}$$

$$= \lim_{h\to 0} \frac{-(2-h)^2 \cos\left(\frac{\pi}{2-h}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2 \sin\left[\frac{\pi}{2} - \frac{\pi}{2-h}\right]}{h}$$

$$= \lim_{h\to 0} \frac{(2-h)^2}{h} \cdot \sin\left[\frac{-\pi h}{2(2-h)}\right]$$

$$= -\lim_{h\to 0} \frac{(2-h)^2}{\frac{\pi h}{2(2-h)}} \cdot \sin\frac{\pi h}{2(2-h)} \times \frac{\pi}{2(2-h)} = -\pi$$

48. **Sol. (D)**

Given equation is $z^2 + z + 1 - a = 0$

Clearly this equation do not have real roots if

$D < 0$

$\Rightarrow 1 - 4(1-a) < 0$

$\Rightarrow 4a < 3$

$a < \frac{3}{4}$

49. **Sol. (C)**

Equation of ellipse is $(y+2)(y-2)+\lambda(x+3)(x-3)=0$

It passes through $(0,4)\Rightarrow \lambda=\frac{4}{3}$

Equation of ellipse is $\frac{x^2}{12}+\frac{y^2}{16}=1$

$e=\frac{1}{2}$.

Alternate

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as it is passing through $(0,4)$ and $(3,2)$.

So, $b^2=16$ and $\frac{9}{a^2}+\frac{4}{16}=1$

$\Rightarrow a^2=12$

So, $12=16(1-e^2)$

$\Rightarrow e=1/2$.

50.

50. **Sol. (B)**

$f(x)=2x^3-15x^2+36x+1$

$f'(x)=6x^2-30x+36$

$=6(x^2-5x+6)$

$=6(x-2)(x-3)$

$f(x)$ is increasing in $[0,2]$ and decreasing in $[2,3]$

$f(x)$ is many one

$f(0)=1$

$f(2)=29$

$f(3)=28$

Range is $[1,29]$

Hence, $f(x)$ is many-one-onto

51. **Sol. (A,B)**

Slope of tangent $=2$

The tangents are $y=2x\pm\sqrt{9\times4-4}$

i.e., $2x-y=\pm4\sqrt{2}$

$\Rightarrow \frac{x}{2\sqrt{2}}-\frac{y}{4\sqrt{2}}=1$ and $-\frac{x}{2\sqrt{2}}+\frac{y}{4\sqrt{2}}=1$

Comparing it with $\frac{xx_1}{9}-\frac{yy_1}{4}=1$

We get point of contact as $\left(\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

Alternate:

Equation of tangent at $P(\theta)$ is $\left(\frac{\sec\theta}{3}\right)\times-\left(\frac{\tan\theta}{2}\right)y=1$

$$\Rightarrow \text{Slope} = \frac{2\sec\theta}{3\tan\theta} = 2$$

$$\Rightarrow \sin\theta = \frac{1}{3}$$

$$\Rightarrow \text{points are } \left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{and} \left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

52. **Sol. (A,C,D)**

$$2\cos\theta(1-\sin\varphi) = \frac{2\sin^2\theta}{\sin\theta}\cos\varphi - 1 = 2\sin\theta\cos\varphi - 1$$

$$2\cos\theta - 2\cos\theta\sin\varphi = 2\sin\theta\cos\varphi - 1$$

$$2\cos\theta + 1 = 2\sin(\theta+\varphi)$$

$$\tan(2\pi-\theta) > 0 \Rightarrow \tan\theta < 0 \text{ and } -1 < \sin\theta < -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$$

$$\frac{1}{2} < \sin(\theta+\varphi) < 1$$

$$\Rightarrow 2\pi + \frac{\pi}{6} < \theta + \varphi < \frac{5\pi}{6} + 2\pi$$

$$2\pi + \frac{\pi}{6} - \theta_{\max} < \varphi < 2\pi + \frac{5\pi}{6} - \theta_{\max}$$

$$\frac{\pi}{2} < \varphi < \frac{4\pi}{3},$$

53. **Sol. (A, D)**

$$\frac{dy}{dx} - y\tan x = 2x\sec x$$

$$\cos x\frac{dy}{dx} + (-\sin x)y = 2x$$

$$\frac{d}{dx}(y\cos x) = 2x$$

$$y(x)\cos x = x^2 + c, \text{where } c = 0 \text{ since } y(0) = 0$$

$$\text{when } x = \frac{\pi}{4}, y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}, \text{when } x = \frac{\pi}{3}, y\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9}$$

$$\text{when } x = \frac{\pi}{4}, y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}} + \frac{\pi}{\sqrt{2}}$$

$$\text{when } x = \frac{\pi}{3}, y'\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{3\sqrt{3}} + \frac{4\pi}{\sqrt{3}}$$

54. **Sol. (B, D)**

$$P(X_1) = \frac{1}{2}, P(X_2) = \frac{1}{4}, P(X_3) = \frac{1}{4}$$

$$P(X) = P(X_1 \cap X_2 \cap X_3^C) + P(X_1 \cap X_2^C \cap X_3) + P(X_1^C \cap X_2 \cap X_3) + P(X_1 \cap X_2 \cap X_3) = \frac{1}{4}$$

(A) $P(X_1^C/X) = \dfrac{P(X \cap X_1^C)}{P(X)} = \dfrac{\frac{1}{32}}{\frac{1}{4}} = \dfrac{1}{8}$

(B) $P[\text{exactly two engines of the ship are functioning} | X] = \dfrac{\frac{7}{32}}{\frac{1}{4}} = \dfrac{7}{8}$

(C) $P\left(\dfrac{X}{X_2}\right) = \dfrac{\frac{5}{32}}{\frac{1}{4}} = \dfrac{5}{8}$

(D) $P\left(\dfrac{X}{X_1}\right) = \dfrac{\frac{7}{32}}{\frac{1}{2}} = \dfrac{7}{16}$

55. **Sol. (A, B, D)**

$S > \frac{1}{e}$ (As area of rectangle $OCDS = 1/e$)

Since $e^{-x^2} \geq e^{-x} \forall x \in [0,1]$

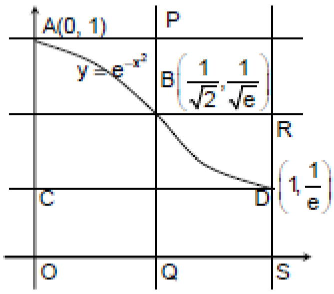
$\Rightarrow S > \int_0^1 e^{-x} dx = \left(1 - \frac{1}{e}\right)$

Area of rectangle $OAPQ +$ Area of rectangle $QBRS > S$

$S < \frac{1}{\sqrt{2}}(1) + \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$.

Since $\frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right) < 1 - \frac{1}{e}$

Hence, (C) is incorrect.



56. **Sol. (3)**

As, $\left|\vec{a}-\vec{b}\right|^2+\left|\vec{b}-\vec{c}\right|^2+\left|\vec{c}-\vec{a}\right|^2=3\left(\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2\right)-\left|\vec{a}+\vec{b}+\vec{c}\right|^2$

$\Rightarrow 3\times 3-\left|\vec{a}+\vec{b}+\vec{c}\right|=9$

$\Rightarrow \left|\vec{a}+\vec{b}+\vec{c}\right|=0 \Rightarrow \vec{a}+\vec{b}+\vec{c}=0$

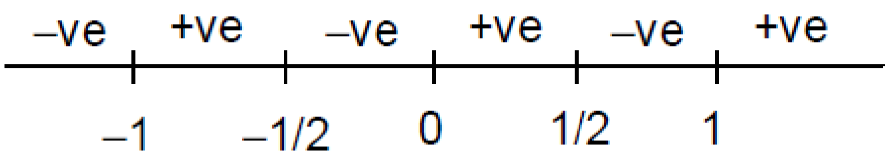
$\Rightarrow \vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow \left|2\vec{a}+5\left(\vec{b}+\vec{c}\right)\right|=\left|-3\vec{a}\right|=3\left|\vec{a}\right|=3.$

57. **Sol. (5)**

$$f'(x)=\frac{|x|}{x}+\frac{\left|x^2-1\right|}{x^2-1}\cdot(2x)$$

$$=\begin{cases} 2x-1 & , \quad x<-1 \\ -(2x+1) & , \quad -1<x<0 \\ 1-2x & , \quad 0<x<1 \\ 2x+1 & , \quad x>1 \end{cases}$$



So, $f'(x)$ changes sign at points

$$x=-1,-\frac{1}{2},0,\frac{1}{2},1$$

so, total number of points of local maximum or minimum is 5.

58. **Sol. (4)**

The parabola is $x=2t^2, y=4t$

Solving it with the circle we get :

$4t^2+16t^2-4t^2-16t^2=0$

$\Rightarrow t^4+3t^2-4t=0 \Rightarrow t=0,1$

so, the points P and Q are $(0,0)$ and $(2,4)$ which are also diametrically opposite points on the circle. The focus is $S\equiv(2,0)$

The area of $\Delta PQS=\frac{1}{2}\times2\times4=4$.

59. **Sol. (9)**

Let $p'(x)=k(x-1)(x-3)$

$\Rightarrow p(x)=k\left(\frac{x^3}{3}-2x^2+3x\right)+c$

Now, $p(1)=6\Rightarrow\frac{4}{3}k+c=6$

also, $p(3)=2\Rightarrow 2c=2$

so, $k=3$, so, $p'(0)=3k=9$.

60. **Sol. (4)**

Let $\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\ldots}}} = y$

So, $4-\frac{1}{3\sqrt{2}}y = y^2 \quad (y>0)$

$\Rightarrow y^2+\frac{1}{3\sqrt{2}}y-4=0 \Rightarrow y=\frac{8}{3\sqrt{2}}$

so, the required value is $6+\log_{3/2}\left(\frac{1}{3\sqrt{2}}\times\frac{8}{3\sqrt{2}}\right)$

$6+\log_{3/1}\frac{4}{9}=6-2=4$.