JEE ADVANCED-2017

MATHEMATICS

37. **Ans. (B)**

Sol.

$f''(x) > 0$ for all $x \in R$, $f(1/2) = 1/2$, $f(1) = 1$

$\Rightarrow f'(x)$ increases

Let $g(x) = f(x) - x$, $x \in [1/2, 1]$

Then $g'(x) = 0$ has at least one real root in $(1/2, 1)$

$f'(x) = 1$ has at least one real root in $(1/2, 1)$

Hence $f'(x)$ increases $\Rightarrow f'(1) > 1$

38. **Ans. (B)**

Sol.

$$\frac{dy}{dx} = \frac{\left(\sqrt{4+\sqrt{9+x}}\right)^{-1}}{8\sqrt{x}\sqrt{9+\sqrt{x}}}$$

$$dy = \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{\sqrt{9+\sqrt{x}}} \cdot \frac{1}{8\sqrt{x}} dx$$

Let $4+\sqrt{9+\sqrt{x}}=t \Rightarrow \frac{1}{2\sqrt{9+\sqrt{x}}}\times\frac{1}{2\sqrt{x}}dx=dt$

$$\int dy=\int\frac{1}{\sqrt{t}}\cdot\frac{1}{2}dt$$

$$y=\sqrt{t}+c$$

$$y=\sqrt{4+\sqrt{9+\sqrt{x}}}+c$$

at $x=0, y=\sqrt{7}$

$$\Rightarrow \sqrt{7}=\sqrt{7}+c \Rightarrow c=0$$

$$y=\sqrt{4+\sqrt{9+\sqrt{x}}}$$

at $x=256 \Rightarrow y=\sqrt{4+\sqrt{9+\sqrt{256}}}=3$

39. **Ans. (A)**

Sol.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+i^2=5$$

Case-1: Five $(1's)$ and four $(0's)$

$${}^9C_5=126$$

Case-II: One (2) and one (1)

$${}^9C_2\times 2!=72$$

$$\therefore Total=198$$

40. **Ans. (C)**

Sol.

$x+y+z=10$

Total number of non-negetive solution $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Now let $z=2n$

$x+y+2n=10$; $n\geq 0$

Total number of non-negative solution $=11+9+7+5+3+14=36$

Required probability $=\frac{36}{66}=\frac{6}{11}$

41. **Ans. (C)**

Sol.

$N_1 = {}^5C_1.{}^4C_4 = 5$

$N_2 = {}^5C_2.{}^4C_3 = 40$

$N_3 = {}^5C_3.{}^4C_2 = 60$

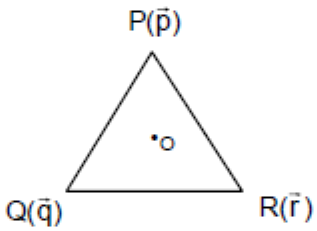
$N_4 = {}^5C_4.{}^4C_1 = 20$

$N_5 = {}^5C_5.{}^4C_0 = 1$

$\therefore Total = 126$

42. **Ans. (B)**

Sol.



$\vec{p}.\vec{q}+\vec{r}.\vec{s}=\vec{r}.\vec{p}+\vec{q}.\vec{s}=\vec{q}.\vec{r}+\vec{p}.\vec{s}$

$\Rightarrow \vec{p}.\left(\vec{q}-\vec{r}\right)-\vec{s}.\left(\vec{q}-\vec{r}\right)=0 \Rightarrow \overrightarrow{PS}.\overrightarrow{QR}=0$

$\Rightarrow S$ is orthocentre of the triangle

43. **Ans. (D)**

Sol.

Let plane be

$a(x-1)+b(y-1)+c(z-1)=0$

Now direction ratio of its normal $=\begin{vmatrix} \hat{i} & j & k \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix}=\hat{i}\left(-14\right)-j\left(2\right)+k\left(-15\right)$

So,

$-14\left(x-1\right)-2\left(y-1\right)-15\left(z-1\right)=0$
$14x+2y+15z=31$

44. **Ans. (A, C)**

Sol.

$f'(x) - 2f(x) > 0$

$\Rightarrow \quad \frac{d}{dx}\left(f(x).e^{-2x}\right) > 0 \quad \Rightarrow \quad g(x) = f(x).e^{-2x}$ is an increasing function.

for $x > 0, \quad g(x) > g(0)$

$\Rightarrow \quad f(x).e^{-2x} > 1 \quad \Rightarrow \quad f(x) > e^{2x}$

Now $f'(x) > 2f(x) > 2.e^{2x}$

$\therefore f(x)$ is an increasing function

45. **Ans. (BD)**

Sol.

Put $x - k = p$

$$I = \sum_{K=1}^{98} \int_0^1 \frac{k+1}{(k+p)(k+p+1)} dp$$

$$I > \sum_{K=1}^{98} \int_0^1 \frac{k+1}{(k+p+1)^2} dp$$

$$I > \sum_{K=1}^{98} (k+1)\left(\frac{-1}{(k+p+1)}\right)_0^1$$

$$I > \sum_{K=1}^{98} (k+1)\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$$

$$I > \sum_{K=1}^{98} \frac{1}{k+2} = \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{100}$$

$$I > \frac{1}{100} + \ldots\ldots\ldots + \frac{1}{100} = \frac{98}{100}$$

$$I > \frac{49}{50}$$

$$\sum_{k=1}^{98} \int_{k}^{k+1} \frac{k+1}{x(x+1)} dx$$

$$\Rightarrow \frac{k+1}{x(x+1)} < \frac{k+1}{x(k+1)} \quad (\because \text{ least value of } x+1 \text{ is } k+1\ )$$

$$\Rightarrow \frac{k+1}{x(x+1)} < \frac{1}{x}$$

$$\Rightarrow I < \sum_{k=1}^{98} \int_{k}^{k+1} \frac{1}{x} dx$$

$$\Rightarrow I < \sum_{k=1}^{98} \ell n(k+1) - \ell nk \Rightarrow I < \ell n 99$$

46. **Ans. (AC)**

Sol.

$$y = x^3$$

$$\int_0^1 (x - x^3)\, dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\int_0^{\alpha} (x - x^3) dx = \frac{1}{8}$$

$$4\alpha^2 - 2\alpha^4 = 1$$

$2\alpha^4 - 4\alpha^2 + 1 = 0$

$2t^2 - 4t + 1 = 0$ (taking $t = \alpha^2$)

$t = \dfrac{4 \pm \sqrt{16-8}}{4}$

$t = \dfrac{4 \pm 2\sqrt{2}}{4}$

$t = \alpha^2 = 1 \pm \dfrac{1}{\sqrt{2}}$

$\therefore \alpha^2 = 1 - \dfrac{1}{\sqrt{2}} \Rightarrow \dfrac{1}{2} < \alpha < 1$

47. Ans. (BC)

Sol.

$\cos\alpha = \left(\dfrac{1-a}{1+a}\right)$; $a = \tan^2\dfrac{\alpha}{2}$

$\cos\beta = \left(\dfrac{1-b}{1+b}\right)$; $b = \tan^2\dfrac{\beta}{2}$

$2\left(\left(\dfrac{1-b}{1+b}\right) - \left(\dfrac{1-a}{1+a}\right)\right) + \left(\left(\dfrac{1-a}{1+a}\right)\left(\dfrac{1-b}{1+b}\right)\right) = 1$

$\Rightarrow 2\left((1-b)(1+a) - (1-a)(1+b)\right) + (1-a)(1-b) = (1+a)(1+b)$

$\Rightarrow 2\left(1+a-b-ab-(1+b-a-ab)\right) + 1-a-b+ab = 1+a+b+ab$

$\Rightarrow 4(a-b) = 2(a+b)$

$\Rightarrow 2a - 2b = a + b$

$\Rightarrow a = 3b$

$\tan^2\dfrac{\alpha}{2} = 3\tan^2\dfrac{\beta}{2}$

$\tan\dfrac{\alpha}{2} = \pm\sqrt{3}\tan\left(\dfrac{\beta}{2}\right)$

48. **Ans. (CD)**

Sol.

$$f\left(1^{+}\right) = \lim_{h\to 0}\frac{1-(1+h)(1+h)}{h}\cos\frac{1}{h}$$
$$= \lim_{h\to 0}\frac{1-(1+h)^{2}}{h}\cos\frac{1}{h}$$
$$= \lim_{h\to 0}\frac{-h^{2}-2h}{h}\cos\frac{1}{h}$$
$$= \lim_{h\to 0}(-h-2)\cos\frac{1}{h}$$

$$\Rightarrow \lim_{h\to 0} f\left(1^{+}\right) \textit{doesnot exist}$$

$$f\left(1^{-}\right) = \lim\frac{1-(1-h)(1+h)}{h}\cos\frac{1}{h}$$
$$= \lim_{h\to 0}\frac{1-\left(1-h^{2}\right)}{h}\cos\frac{1}{h}$$
$$= \lim_{h\to 0}\frac{h^{2}}{h}\cos\frac{1}{h}$$
$$= \lim_{h\to 0} h\cos\frac{1}{h} = 0$$

49. **Ans. (BONUS)**

Sol.

$$g(x) = \int_{\sin x}^{\sin(2x)}\sin^{-1}(t)\,dt$$

$$g'(x) = \sin^{-1}(\sin 2x).\cos 2x.2 - \sin^{-1}(\sin x).\cos x$$
$$= 2\cos 2x.\sin^{-1}(\sin 2x) - \cos x.\sin^{-1}(\sin x)$$

$$g'\left(-\frac{\pi}{2}\right) = 2\cos(-\pi)\sin^{-1}(\sin(-\pi)) - \cos\left(-\frac{\pi}{2}\right).\sin^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right) = 0$$
$$g'\left(\frac{\pi}{2}\right) = 2\cos(\pi).\sin^{-1}(\sin(\pi)) - \cos\left(\frac{\pi}{2}\right).\sin^{-1}\left(\sin\left(\frac{\pi}{2}\right)\right) = 0$$

50. **Ans. (BC)**

Sol.

$$f(x) = \begin{vmatrix} \cos 2x & \cos 2x & \sin 2x \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$$

$$= \cos 2x - \cos 2x\left(-\cos^2 x + \sin^2 x\right) + \sin 2x\left(-2\sin x \cos x\right)$$

$$f(x) = \cos 4x + \cos 2x$$

$$\therefore f(x) = 2\cos^2 2x + \cos 2x - 1$$

Let $\cos 2x = t$

$\Rightarrow f(x) = 2t^2 + t - 1$ and $t \in [-1, 1]$

$f(x)$ attains its minima at $t = -\frac{1}{4} \in [-1, 1]$

$$\therefore f(x)\big|_{\min} = \frac{2}{16} - \frac{1}{4} - 1 = \frac{-9}{8}$$

$$\therefore f(x)\big|_{\max} = 2 + 1 - 1 = 2 \ldots\ldots\ldots (when \cos 2x = 1)$$

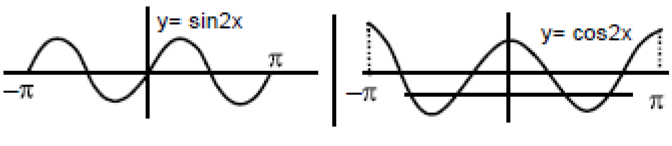
$$f'(x) = -4\sin 4x - 2\sin 2x$$

$$f'(x) = 0 \quad \Rightarrow 4\sin 4x + 2\sin 2x = 0$$

$$\Rightarrow 8\sin 2x \cos 2x + 2\sin 2x = 0$$

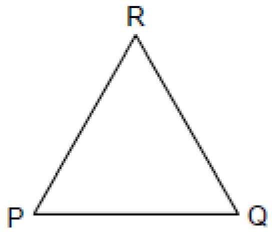
$$\Rightarrow 2\sin 2x\left(4\cos 2x + 1\right) = 0$$

$\Rightarrow \sin 2x = 0$ or $\cos 2x = -\frac{1}{4}$



Hence option (B), (C).

51. **Ans. (A)**



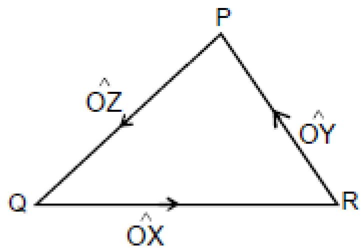
$$\cos(P+Q)+\cos(Q+R)+\cos(R+P)=-\cos R-\cos P-\cos Q$$

if any Δ, max of $\cos P+\cos Q+\cos R=\frac{3}{2}$

So minimum value of the given expression is $-\frac{3}{2}$

52. **Ans. (A)**

Sol.



$$\cos R=-OX.OY$$

$$\Rightarrow |\cos R|=|OX.OY|$$

$$|OX.OY|=|\sin R|=|\sin(\pi-(P+Q))|=|\sin(P+Q)|=\sin(P+Q)$$

53. **Ans. (D)**

Sol.

As α and β are roots of equation $x^2 - x - 1 = 0$, we get:

$\alpha^2 - \alpha - 1 = 0 \quad \Rightarrow \alpha^2 = \alpha + 1$

$\beta^2 - \beta - 1 = 0 \quad \Rightarrow \beta^2 = \beta + 1$

$$\begin{aligned} \therefore a_{11} + a_{10} &= p\alpha^{11} + q\beta^{11} + p\alpha^{10} + q\beta^{10} \\ &= p\alpha^{10}(\alpha + 1) + q\beta^{10}(\beta + 1) \\ &= p\alpha^{10} \times \alpha^2 + q\beta^{10}\beta^2 \\ &= p\alpha^{12} + q\beta^{12} \\ &= a_{12} \end{aligned}$$

54. **Ans. (D)**

Sol.

$a_{n+2} = a_{n+1} + a_n$

$a_4 = a_3 + a_2 = 3a_1 + 2a_0 = 3p\alpha + 3q\beta + 2(p + q)$

As $\alpha = \dfrac{1+\sqrt{5}}{2}, \beta = \dfrac{1-\sqrt{5}}{2}$, we get

$a_4 = 3p\left(\dfrac{1+\sqrt{5}}{2}\right) + 3q\left(\dfrac{1-\sqrt{5}}{2}\right) + 2p + 2q = 28$

$\Rightarrow \left(\dfrac{3p}{2} + \dfrac{3q}{2} + 2p + 2q - 28\right) = 0 \ldots\ldots \text{(i)}$

And $\dfrac{3p}{2} - \dfrac{3q}{2} = 0 \ldots\ldots\ldots \text{(ii)}$

$\Rightarrow p = q$ (from (ii))

$\Rightarrow 7p = 28$

$\Rightarrow p = 4$

$\Rightarrow q = 4$

$\Rightarrow p + 2q = 12$