JEE ADVANCED-2018 (PAPER-1)

MATHEMATICS

1. **Ans. (ABD)**

Sol.

(A) $Arg(-1-i) = -3\frac{\pi}{3}$

(B) $f(t) = Arg(-1+it)$

$$\begin{bmatrix} \pi - \tan^{-1} t & t \geq 0 \\ -(\pi + \tan^{-1} t) & t < 0 \end{bmatrix}$$

If is discontinuous at $t = 0$

(C) $Arg\left(\frac{z_1}{z_2}\right) - Arg\, z_1 + Arg\, z_2$

$Arg\left(\frac{z_1}{z_2}\right) = Arg\, z_1 - Arg\, z_2 + 2n\pi$

so the expression becomes $2n\pi$

(D) $Arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$

If is circle

2. **Ans. (BCD)**

Sol.

$$\cos Q = \frac{100 + 300 - (PR)^2}{2.10.10\sqrt{3}} \Rightarrow \frac{\sqrt{3}}{2} = \frac{100 + 300 - (PR)^2}{2.10.10\sqrt{3}}$$

$$300 = 400 - (PR)^2 \qquad \Rightarrow PR = 10$$

$$\Delta = \frac{1}{2}(PQ)(QR)\sin Q = \frac{1}{2}10.10\sqrt{3} \times \frac{1}{2} = 25\sqrt{3}$$

$$r = \frac{\Delta}{s} = \frac{25\sqrt{3} \times 2}{\left(20 + 10\sqrt{3}\right)} = \frac{50\sqrt{3}}{20 + 10\sqrt{3}} = \frac{5\sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = 5\left(2\sqrt{3} - 3\right) = 10\sqrt{3} - 15$$

$$\text{by sine rule } \frac{10\sqrt{3}}{\sin R} = \frac{10}{\sin Q} \qquad \Rightarrow \angle R = 30$$

$$2(\text{circumradius}) = \frac{PR}{\sin Q} = \frac{10}{1/2} \quad \Rightarrow \text{circumradius} = 10$$

Hence area of circumcircle $= \pi R^2 = 100\pi$

3. Ans. (CD)

Sol. Direction ratio of common line is $n_1 \times n_2$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \hat{i}(3) - \hat{j}(3) + \hat{k}(3) = 3\left(\hat{i} - \hat{j} + \hat{k}\right)$$

(B) $\frac{x - 4/3}{3} = \frac{y - 1/3}{-3} = \frac{z}{3}$

This is $\parallel$ to line of intersection

(C) $\cos\theta = \frac{x_i x_2}{|x_1 \parallel x_2|} = \frac{2 + 2 - 1}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2} \Rightarrow \quad \theta = \frac{\pi}{3}$

(D) $P_3 : x - y + z = l$ satisfy $(4, 2, -2)$

$4 - 2 - 2\hat{i} \quad \Rightarrow x - y + z = 4$

$(211) \perp \Rightarrow \left|\frac{2 - 1 + 1 - 4}{\sqrt{3}}\right| \Rightarrow \quad \frac{2}{\sqrt{3}}$

4. Ans. (ABD)

Sol. $f^2(0)+(f'(0))^2=85 \quad f:R\to[-2,2]$

(A) This is true of every continuous function

(B)

$$|f'(c)|=\left|\frac{f(-4)-f(0)}{-4-0}\right|$$

$$|f'(c)|=\left|\frac{f(-4)-f(0)}{4}\right|$$

$-2\le f(-4)\le 2$

$-2\le f(0)\le 2$

$4\le f(-4)-f(0)\le 4$

$\text{This}\ |f'(c)|\le 1$

(C) $\lim_{x\to\infty} f(x)=1$

Note $f(x)$ should have a bound ∞ which can be concluded by considering

$$f(x)=2\sin\left(\frac{\sqrt{85}x}{2}\right)$$

$$f(x)=\sqrt{85}\cos\left(\frac{\sqrt{85}x}{2}\right)$$

$$f(x)=\sqrt{85}\cos\left(\frac{\sqrt{85}x}{2}\right)$$

$$f^2(0)+\left(f'(0)^2\right)=85$$

and $\lim_{x\to\infty} f(x)$ does not exist

(D) Consider $H(x)=f^2(x)+\left(f'(x)^2\right.$

$H(0)=85$

By (B) choice there exists some x_0 such that $(f'(x_0))^2\le 1$ for some x_0 in $(-4,0)$

Hence

$$H(x_0) = f^2(x_0) + \left(f'(x_0)\right)^2 \le 4 + 1$$
$$H(x_0) \le 5$$

Hence let $p \in (-4, 0)$ for which $H(p) = 5$

(note that we have considered p as largest such negative number)

similarly let q be smallest positive number $\in (0, 4)$ such that $H(q) = 5$

Hence By Rolle's theorem is (p, q)

$H'(c) = 0$ for some $c \in (-4, 4)$ and since $H(x)$ is greater than 5 as we move from $x = p$ to $x = q$ and $f^2(x) \le 4$

$$\Rightarrow \quad \left(f'(x)\right)^2 \ge 1 \; in \; (p, q)$$
$$\text{Thus } H'(c) = 0 \Rightarrow \quad ff' + f'f'' = 0$$
$$\text{so } f + f'' = 0 \text{ and } f' \ne 0$$

5. **Ans. (BC)**

Sol.

$$f'(x) = e^{f(x)-g(x)} g'(x) : f(1) = g(2) = 1$$
$$e^{-f(x)} = e^{-g(x)} + c$$
$$e^{-f(x)} \cdot f'(x) = e^{-g(x)} \cdot g'(x)$$
$$\int d\left(e^{-f(x)}\right) = \int d\left(e^{-g(x)}\right)$$
$$e^{-f(x)} = e^{-g(x)} + c$$
$$x = 1 \; \frac{1}{e} = e^{-g(1)} + c$$
$$x = 2 \; e^{-f(2)} = \frac{1}{e} + c$$

$\therefore \quad g(1) > 1 - \ell n 2$

$e^{-f(2)} = 2e^{-1} - e^{-g(1)}$

$e^{-f(2)} = 2e^{-1} - e^{-g(1)}$

$f(2) > 1 - \ell n 2$

$e^{-1} - e^{-f(2)} = e^{-g(1)} - e^{-1} \Rightarrow \quad e^{-g(1)} + e^{-f(2)} = 2e^{-1}$

$e^{-g(1)} < 2e^{-1}$

$-g(1) < \ell n 2 - 1$

6. **Ans. (BC)**

Sol.

$f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt \quad \Rightarrow f(x) \cdot e^{-x} = (1 - 2x) \cdot e^{-x} + \int_0^x e^{-t} f(t)dt$

$\Rightarrow \quad f'(x)e^{-x} - e^{-x} \cdot f(x) = -2 \cdot e^{-x} + e^{-x} \cdot f(x)$

$\Rightarrow \quad f'(x) - 2f(x) = (2x - 3)$

$I.F. = e^{-2x}$

$\therefore \quad y \cdot e^{-2x} = \int (2x_{|} - 3) \cdot e_{||}^{-2x} dx$

$\Rightarrow \quad y \cdot e^{-2x} = (2x - 3) \cdot \frac{e^{-2x}}{-2} - 2\int \frac{e^{-2x}}{-2} dx \quad \Rightarrow \quad y \cdot e^{-2x} = -\frac{(2x-3)e^{-2x}}{2} - \frac{e^{-2x}}{2} + c$

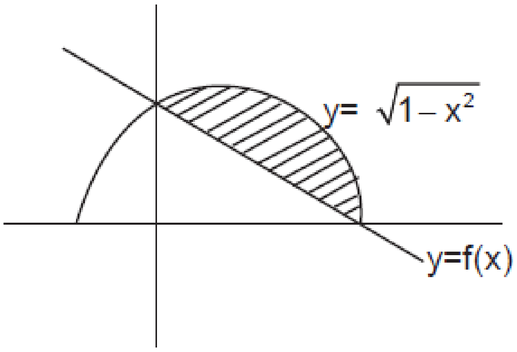
$\Rightarrow \quad y \cdot e^{-2x} = \frac{-(2x-3) - 1}{2} + c.e^{2x} \Rightarrow \quad y = (1 - x) + c.e^{2x}$

$\Rightarrow \quad y = (1 - x) + c.e.^{2x}$

put $x = 0$

$1 = 1 + c \quad \Rightarrow \quad c = 0$

$\therefore \quad y = 1 - x$ which passes through point $(2, -1)$



Now

required area $= \frac{1}{4}\cdot\pi(1)^2 - \frac{1}{2}\cdot 1\cdot 1 = \frac{\pi}{4} - \frac{1}{2}$

7. **Ans. (8)**

Sol.

$$\left(\left(\log_2 9\right)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times \left(\sqrt{7}\right)^{\log_7 4}$$

$$\left(\log_2 9\right)^{2\log_{(\log_2 9)^2}} \cdot (2) = 4.2 = 8$$

8. **Ans. (625)**

Sol. Last two digits are $12, 32, 24, 52, 44$

Number of numbers $= 5\times 5\times 5\times 5 = 625$

9. **Ans. (3748)**

Sol.

$P = \{1, 6, 11, \cdots\}$

$Q = \{9, 16, 23, \cdots\}$

Common terms: $16, 51, 86$

$$t_p = 16 + (p-1)35 = 35p - 19 \le 10086$$

$$\Rightarrow p \le 288.7$$

$$\therefore n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$$

$$= 2018 + 2018 - 288$$

$$= 3748$$

10. Ans. (2)

Sol.

$$\lim_{n\to\infty} \sin^{-1}\left(\sum_{i=1}^{n} x^{i+1} - x\sum_{i=1}^{n}\left(\frac{x}{2}\right)^{i}\right) = \frac{\pi}{2} - \lim_{n\to\infty}\cos^{-1}\left(\sum_{i=1}^{n}\left(-\frac{x}{2}\right)^{i} - \sum_{i=1}^{n}(-x)^{i}\right)$$

$$\left(\frac{x^2}{1-x} - x\frac{\frac{x}{2}}{1-\frac{x}{2}}\right) = \frac{x}{1+x} + \frac{\left(-\frac{x}{2}\right)}{1+\frac{x}{2}}$$

$$\frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{x}{1+x} - \frac{x}{2+x}$$

$$\frac{x^2}{1-x} - \frac{x}{1+x} = \frac{x^2}{2-x} - \frac{x}{2+x}$$

$$\frac{x(1+x)-(1-x)}{1-x^2} = \frac{2x + x^2 - 2 + x}{4-x^2} \text{ or } x = 0$$

$$\frac{x^2+2x-1}{1-x^2} = \frac{x^2+3x-2}{4-x^2}$$

$$\Rightarrow \quad x^3 + 2x^2 + 5x - 2 = 0$$

Let $f(x) = x^3 + 2x^2 + 5x - 2 = 0$

$f'(x) > 0$

$f(0) = -2$ and $f(1/2) = 9/8$ so one root in $\left(0, \frac{1}{2}\right)$

$\Rightarrow$ 2 roots

11. **Ans. (1)**

Sol.

$$y_n = \left(\frac{n+1}{n}\frac{n+2}{n}\cdots\frac{n+n}{n}\right)^{\frac{1}{n}}$$

$$\log L = \lim_{x\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$= \int_0^1 \log(1+x)\,dx = \int_1^2 \log x\,dx = \left|x\log x - x\right|_1^2 = \log 2 = \log\frac{4}{e}$$

$$\Rightarrow \quad L = \frac{4}{e} \Rightarrow \quad [L] = 1$$

12. **Ans. (3)**

Sol.

$$\vec{c} = x\vec{a} + y\vec{b} + \vec{a}\times\vec{b} \ \& \ \vec{a}.\vec{b} = 0$$

$$\vec{a}\wedge\vec{c} = \vec{b}\wedge\vec{c} = \alpha$$

$$\vec{c}\cdot\vec{a} = \vec{c}\cdot\vec{b} = 2\cos\alpha \quad \Rightarrow \quad x = y = 2\cos\alpha$$

$$|\vec{c}|^2\ x^2 + y^2 + |\vec{a}\times\vec{b}|^2 = 2\left(4\cos^2\alpha\right) + 1 - 0$$

$$4 = 8\cos^2\alpha + 1 \quad \Rightarrow \quad 8\cos^2\alpha = 3$$

13. **Ans. (0.5)**

Sol.

$$\sqrt{3}a\cos x + 2b\sin x = c \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sqrt{3}a\left(\frac{1-t^2}{1+t^2}\right) + 2b\left(\frac{2t}{1+t^2}\right) = c, \text{ where } t = \tan\frac{x}{2}$$

$$\sqrt{3}a\left(1-t^2\right)+4bt=c\left(1+t^2\right)$$

$$t^2\left(c+\sqrt{3}a\right)-4bt+c-\sqrt{3}a=0$$

$$\frac{\alpha+\beta}{2}=\frac{\pi}{6}$$

$$\tan\left(\frac{\alpha+\beta}{2}\right)=\frac{1}{\sqrt{3}}$$

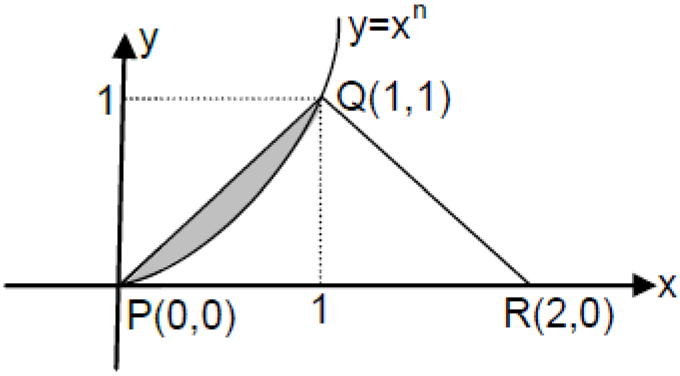
$$\Rightarrow \quad \frac{t_1+t_2}{1-t_1t_2}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{4b}{c+\sqrt{3}a-c+\sqrt{3}a}=\frac{1}{\sqrt{3}}$$

$$\frac{b}{a}=\frac{1}{2}$$

14. Ans. (4)

Sol.



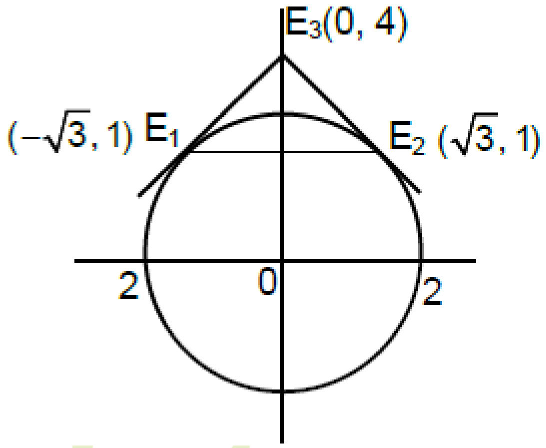
$$\int_0^1\left(x-x^n\right)dx=\frac{3}{10}\left(\frac{1}{2}\times 2\times 1\right) \quad \Rightarrow \quad \left|\frac{x^2}{2}-\frac{x^{n+1}}{n+1}\right|_0^1=\frac{3}{10} \quad \Rightarrow \quad \frac{1}{2}-\frac{1}{n+1}=\frac{3}{10}$$

$$\Rightarrow \frac{1}{n+1}=\frac{1}{2}-\frac{3}{10}=\frac{1}{5} \qquad \Rightarrow \quad n=4$$

15. Ans. (A)

Sol. Tangent at E_1 and E_2 are $-\sqrt{3}x+y=4$ and $\sqrt{3}x+y=4$

They intersect at $E_3(0,4)$



$F_1(1,\sqrt{3}), F_2(1,-\sqrt{3}), F_3(4,0)$

$G_1(0,2), G_2(2,0), G_3(2,2)$

E_3, F_3, G_3 lie on line $x + y = 4$

16. **Ans. (D)**

Sol.

Let $P(2\cos\theta, 2\sin\theta)$

Tan get is $x + y\sin\theta = 2$

$M\left(\frac{2}{\cos\theta}\right), N\left(0, \frac{2}{\cos\theta}\right)$

$x = \frac{1}{\cos\theta}$ and $y = \frac{1}{\sin\theta} \Rightarrow \frac{1}{x^2} + \frac{1}{y^2} = 1 \Rightarrow x^2 + y^2 = x^2y^2$

17. **Ans. (A)**

Sol.

$$\text{Probability} = \frac{4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)}{5!} = \frac{9}{120} = \frac{3}{40}$$

18. **Ans. (C)**

Sol.

Total cases = 5!

favorable ways = 14

$$\left.\begin{matrix}\underline{1} & \underline{3} & \underline{5} & \underline{2} & \underline{4} \\ \underline{1} & \underline{4} & \underline{2} & \underline{5} & \underline{3}\end{matrix}\right\} \to 2$$

$5 \to 2$

$\underline{2} \quad \underline{4} \quad \underline{1} \quad \cdots \quad \cdots \quad \to \quad 2$

$\underline{2} \quad \underline{5} \quad \underline{3} \quad \underline{1} \quad \underline{4} \quad \to \quad 1$

$4 \to 3$

$$\left.\begin{matrix}\underline{3} & \underline{1} & \underline{5} & \underline{2} & \underline{4} \\ \underline{3} & \underline{1} & \underline{4} & \underline{2} & \underline{5}\end{matrix}\right\} \to 2$$

$= 14$

$$\text{Probability} = \frac{14}{120}$$