JEE ADVANCED-2018 (PAPER-2)

MATHEMATICS

1. **Ans. (ABD)**

Sol.

$$f_n(x)=\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$$

$$f_n(x)=\tan^{-1}(x+n)-\tan^{-1}(x)\quad\Rightarrow\quad f_n'(x)=\frac{1}{1+(x+n)^2}-\frac{1}{1+x^2}$$

$$f_n(0)=\tan^{-1}(n)\Rightarrow\quad \tan^2\left(\tan^{-1}n\right)=n^2$$

(A) $\sum_{j=1}^{5}\tan^2\left(f_j(0)\right)=\sum_{j=1}^{5}j^2=\frac{5.6.11}{6}=55$

(B)

$$f_n'(0)=\frac{1}{1+n^2}-1\quad\Rightarrow 1+f_n(0)=\frac{1}{1+n^2}$$

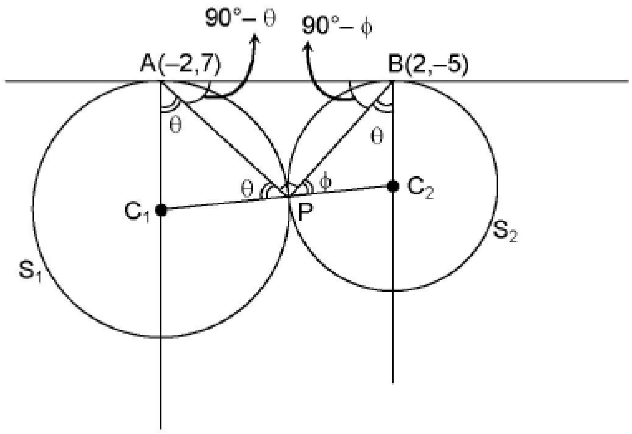
$$\sec^2\left(f_n(0)\right)=\sec^2\left(\tan^{-1}(n)\right)=1+n^2.$$

$$\text{Hence}\left(1+f_n'(0)\right).\sec^2\left(f_n(0)\right)=\left(\frac{1}{1+n^2}\right)\left(1+n^2\right)=1$$

$$\text{so}\sum_{i=1}^{10}\left(1+f_i'(0)\right)\sec^2\left(f_i(0)\right)=\sum_{i=1}^{10}1=10$$

$$\lim_{x\to\infty}\tan\left(fn(x)\right)=0\qquad\&\qquad\lim_{x\to\infty}\sec^2\left(f_n(x)\right)=1$$

2. Ans. (AD)



Sol. Note that $\angle APB = \frac{\pi}{2}$, hence locus of P is $(x+2)(x-2)+(y-7)(y+5)=0$

$x^2 + y^2 - 2y - 39 = 0 \quad \cdots E_1$

Locus of mid-points of chords passing through $(1,1)$ is

$h + K - (1+k) = h^2 + k^2 - 2K$

$\Rightarrow h^2 + K^2 - 2K - h + 1 = 0$

Hence E_2 is $x^2 + y^2 - x - 2y + 1 = 0$

3. Ans. (ACD)

Sol. $\Delta = 0$ so for at least one solutions $\Delta_1 = \Delta_2 = \Delta_3 = 0 \Rightarrow b_1 + 7b_2 = 13b_3 \quad \cdots (i)$

option $(A)\ \Delta \neq 0 \quad \Rightarrow$ unique solution $\Rightarrow$ option (A) is correct

option $(D)\ \Delta \neq 0 \quad \Rightarrow$ unique solution $\Rightarrow$ option (B) is correct

option $(C)\ \Delta \neq 0 \quad \Rightarrow$ equations are $x - 2y + 5z = -b_1$

$x - 2y + 5z = \frac{b_2}{2}$

$x - 2y + 5z = b_2$

There planes are parallel so they must be coincident

$\Rightarrow \quad -b_1 = \frac{b_2}{2} = b_3$ which satisfies equation (1) for all $b_1, b_2, b_3 \Rightarrow$ option (C) is correct.

option (B) $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 5 & 2 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 0, \text{Also } \Delta_1 = 0$

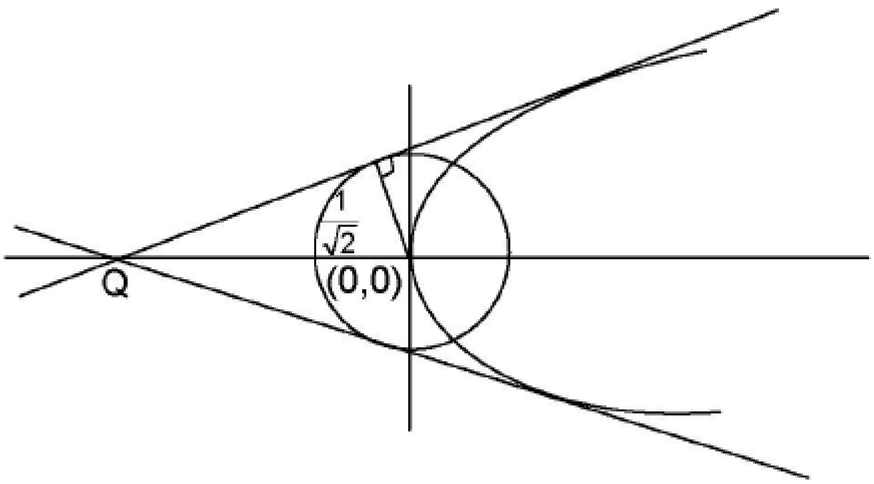
For infinite solutions, Δ_2 and Δ_3 must be 0

$\Rightarrow \begin{vmatrix} 1 & b_1 & 1 \\ 5 & b_2 & 2 \\ 2 & b_3 & 1 \end{vmatrix} = 0 \Rightarrow -b_1 - b_2 + 3b_3 = 0$ which does bot satisfy (i) for all b_1, b_2, b_3

so option(s) wrong

4. **Ans. (AC)**

Sol.



Let equation of common tangent is $y = mx + \frac{1}{m}$

$\therefore \left| \frac{0 + 0 + \frac{1}{m}}{\sqrt{1 + m^2}} \right| = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

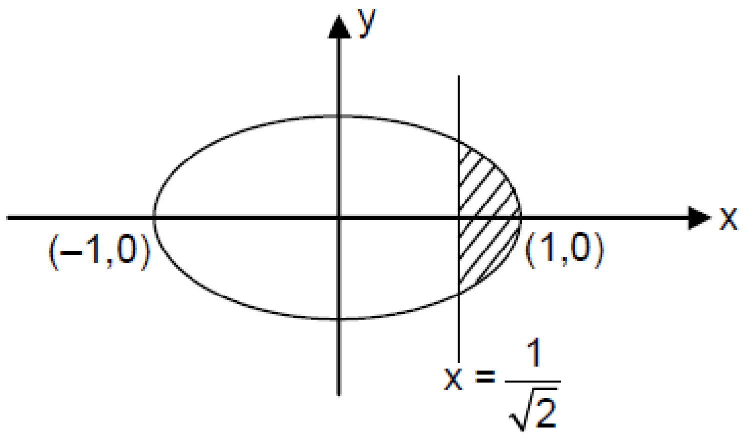
$\Rightarrow \quad m = \pm 1$

Equation of common tangents are $y = x + 1$ & $y = -x - 1$

point Q is $(-1, 0)$

$\therefore$ Equation of ellipse is $\dfrac{x^2}{1} + \dfrac{y^2}{1/2} = 1$

(A) $e = \sqrt{1 - \dfrac{1}{2}} = \dfrac{1}{\sqrt{2}}$ & $LR = \dfrac{2b^2}{a} = 1$



(C)

$$Area \quad 2 \cdot \int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{2}} \cdot \sqrt{1 - x^2}\, dx$$

$$= \sqrt{2}\left[\frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x\right]_{1/\sqrt{2}}^{1}$$

$$= \sqrt{2}\left[\frac{\pi}{4} - \left(\frac{1}{4} + \frac{\pi}{8}\right)\right] = \sqrt{2}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi - 2}{4\sqrt{2}}$$

5. **Ans. (ACD)**

Sol.

$$sz + t\bar{z} + r = 0, \bar{z} = x - iy$$

$$\overline{sz} + \bar{t}z + r = 0$$

$$(1) + (2)$$

$$(t + \bar{s})\bar{z} + (s + \bar{t})z + (r + \bar{r}) = 0$$

$$(t - \bar{s})\bar{z} + (s - \bar{t})z + (r - \bar{r}) = 0$$

For unique solution

$$\frac{t+\bar{s}}{t-\bar{s}} \neq \frac{s+\bar{t}}{s-\bar{t}}$$

On solving the above equation we get

$$|t| \neq |s|$$

$\therefore$ option (A) is correct

Lines overlap if

$$\frac{t+\bar{s}}{t-\bar{s}} = \frac{\bar{t}+s}{s-\bar{t}} = \frac{r+\bar{r}}{r-\bar{r}}$$

$$|t| = |s| \quad \bar{t}r - \overline{tr} + sr - s\bar{r} = sr + s\bar{r} - \bar{t}r - \overline{tr}$$

$$2\,t\bar{r} = 2\,s\bar{r}$$

$$\bar{t}r = s\bar{r}$$

$$\therefore |\bar{t}\,||\,r| = |s\,||\,r|$$

$$\therefore |t| = |s|$$

$\therefore$ If $|t| = |s|$, lines will be parallel for sure but it may not be coincident

For option (C) if element of set L represent line, then this line and given circle can have maximum two common points so option (C) is correct

6. **Ans. (BCD)**

Sol.

$$\lim_{t\to x} \frac{f(x)\sin t - f(t)\sin x}{t-x} = \sin^2 x$$

$$\frac{f(x)\cos x - f'(x)\sin x}{\sin^2 x} = 1$$

$$-d\left(\frac{f(x)}{\sin x}\right) = 1$$

$$\frac{f(x)}{\sin x} = -x + c \because \quad f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12} \Rightarrow \quad c = 0 \Rightarrow \quad f(x) = -x\sin x$$

(A) $f(x) + f'(x) = -2\cos x$

$$f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right) = 0$$

(B) $f\left(\frac{\pi}{4}\right)=\left(-\frac{\pi}{4\sqrt{2}}\right)$

(C) $f(x)$ is continuous and differentiable and $f(0)=f(x)=0$ using $LMVT\ f'(c)=0$ for some $x\in(0,\pi)$

(D)

$$g(x)=-x\sin x+x^{2}-\frac{x^{4}}{6}$$

$$g'(x)=f'(x)+2x-\frac{2x^{3}}{3}$$

$$g''(x)=f''(x)+2x-2x^{2}$$

$$g'''(x)=3\sin x+x\cos x-4x=3(\sin x-x)+x(\cos x-1)$$

$$\Rightarrow\quad g'''(x)<0\quad\Rightarrow\quad g''(x)\text{ is decreasing}$$

$$\text{for } x>0\quad g''(x)\quad<g''(0)\Rightarrow\quad g''(x)<0$$

hence $g'(x)$ is decreasing

for $x>0\quad g'(x)<g'(0)\quad\Rightarrow\quad g'(x)<0$

hence $g(x)<0$

for $x>0\quad g(x)<g(0)\quad\Rightarrow\quad g(x)<0$

Hence $f(x)<\frac{x^{4}}{6}-x^{2}\forall z\in(0,\pi)$

7. **Ans. (2)**

Sol.

$$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$$

$$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$$

Put

$$\frac{1-x}{1+x}=t \quad \Rightarrow \quad \frac{-2dx}{(1+x)^2}=dt$$

$$I=\int_0^{\frac{1}{3}} \frac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}}=\frac{-\left(1+\sqrt{3}\right)}{2}\times\left|\frac{-2}{\sqrt{t}}\right|_1^{1/3}=\left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right)=2$$

8. **Ans. (4)**

Sol.

$$\det(P)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=a_1\left(b_2c_3-b_3c_2\right)-a_2\left(b_1c_3-b_3c_1\right)+a_3\left(b_1c_2-b_2c_1\right)\le 6$$

value can be 6 only if $a_1=1, a_2=-1, a_3=1, b_2c_3=b_1c_3=b_1c_2=1, b_3c_2=b_3c_1=b_2c_1=-1$

$$\Rightarrow \quad \left(b_2c_3\right)\left(b_3c_1\right)\left(b_1c_2\right)=-1 \quad \& \quad \left(b_1c_3\right)\left(b_3c_2\right)\left(b_2c_1\right)=1$$

i.e. $b_1b_2b_3c_1c_2c_3=1 \text{ and } -1$

hence not possible

Similar contradiction occurs when
$a_1=1, a_2=1, a_3=1, b_2c_2=b_3c_1=b_1c_2=1 b_3c_2=b_1c_3=b_1c_2=-1$

Now for value to be 5 one the terms must be zero but that will make 2 terms zero which means answer

cannot be 5

Now $\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 4$ Hence max value $= 4$

9. **Ans. (119)**

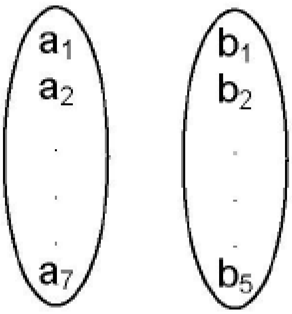
Sol.

$n(X) = 5$

$n(Y) = 7$

$\alpha \to$ Number of one-one function $= {}^7C_5 \times 5! = 21 \times 120 = 2520$

$\beta \to$ Number of onto function to Y to X



$1,1,1,1,3 \quad 1,1,1,2,2$

$$\frac{7!}{3!4!} \times 5! + \frac{7!}{(2!)^3 3!} \times 5! = \left({}^7C_3 + 3 \cdot {}^7C_3\right)5! = 4 \times {}^7C_3 \times 5!$$

$$\frac{\beta - \alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$$

10. **Ans. (0.4)**

Sol.

$$\frac{dy}{dx} = (5y+2)(5y-2)$$

$$\frac{1}{25}\int \frac{dy}{\left(y+\frac{2}{5}\right)\left(y-\frac{2}{5}\right)} = \int dx$$

$$\frac{1}{25}\cdot\frac{5}{4}\ell n\left|\frac{y-\frac{2}{5}}{y+\frac{2}{5}}\right| = x + c$$

$$\frac{1}{20}\ell n\left|\frac{5y-2}{5y+2}\right| = x + c$$

$$\text{at } x = 0, y = 0 \Rightarrow c = 0$$

$$\text{Hence } \frac{2-5y}{2+5y} = e^{20x}$$

$$\frac{2-5y}{2+5y} = e^{20x}, \lim_{x\to\infty} e^{20x} = 0 \quad \Rightarrow \quad \lim_{x\to\infty} y = \frac{2}{5} = 0.4$$

11. **Ans. (2)**

Sol. $f(x+y) = f(x)\cdot f'(y) + f'(x)\cdot f(y)$

substituting $x = y = 0$, we get

$$f(0) = 2f'(0) \quad \Rightarrow \quad f'(0) = \frac{1}{2}$$

Now substituting $y = 0$

$$f(x) = f(x)\cdot f'(0) + f'(x)\cdot f(0)$$

$$\Rightarrow \quad f'(x) = \frac{f(x)}{2}$$

$$\Rightarrow \quad f(x) = \lambda e^{x/2} \quad \Rightarrow \quad f(x) = e^{x/2} \left(\text{as } f(0) = 1\right)$$

Now $\ln(f(x)) = \frac{x}{2} \Rightarrow \ln(f(4)) = 2$

12. **Ans. (8)**

Sol.

$P(\alpha, \beta, \gamma)$

$R(\alpha, \beta, -\gamma)$

Q

$\frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{0} = \frac{-2(\alpha+\beta-3)}{2}$

$x = 3-\beta, y = 3-\alpha, z = \gamma$

$Q(3-\beta, 3-\alpha, \gamma)$ lies on z-axis

$\therefore \quad \beta = 3, \alpha = 3$

$P(3,3,\gamma)$ distance from x-axis is 5

$9+\gamma^2 = 25$

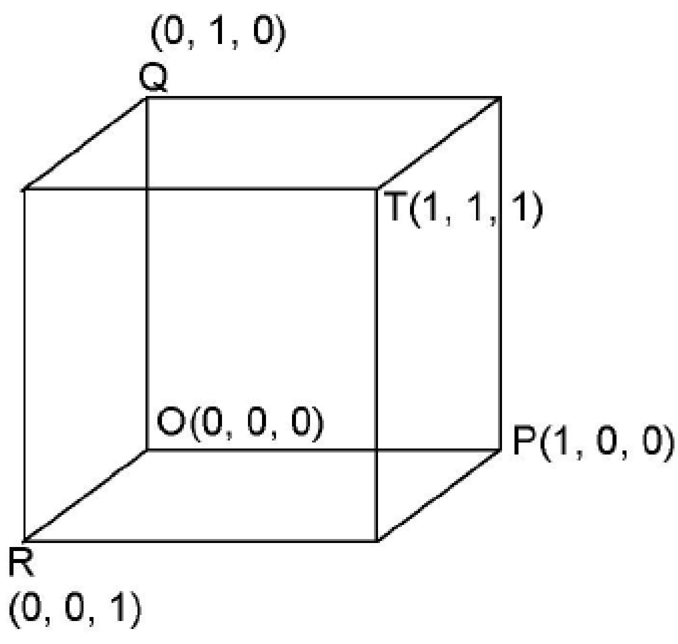
$\gamma^2 = 16 \Rightarrow \gamma = 4$

$P(3,3,4) \quad \therefore \quad PR = 8$

$R(3,3,-4)$

13. **Ans. (0.5)**

Sol.



point $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

point $T\ (1,1,1)$

$\vec{p} = \overline{SP} = \frac{\hat{i}-\hat{j}-\hat{k}}{2}$

$\vec{q} = \overline{SQ} = \frac{-\hat{i}-\hat{j}-\hat{k}}{2}$

$\vec{r} = \overline{SR} = \frac{-\hat{i}-\hat{j}+\hat{k}}{2}$

$\vec{t} = \overline{ST} = \frac{\hat{i}+\hat{j}+\hat{k}}{2}$

Now

$$\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix}\times\frac{1}{4} = \frac{1}{4}\left(2\hat{i}+2\hat{j}\right) = \frac{i+\hat{j}}{2}$$

$$\vec{r}\times\vec{t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}\times\frac{1}{4} = \frac{-2\hat{i}+2\hat{j}}{4} = \frac{\hat{i}+\hat{j}}{2}$$

Now $\left(\vec{p}\times\vec{q}\right)\times\left(\vec{r}\times\vec{t}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix}\times\frac{1}{4} = \frac{\hat{k}}{2} \Rightarrow \left|\left(\vec{p}\times\vec{q}\right)\times\left(\vec{r}\times\vec{t}\right)\right| = \frac{1}{2} = 0.5$

14. Ans. (646)

Sol. $X = \sum_{r=1}^{10} r\cdot{}^{10}C_r\cdot{}^{10}C_r = 10\cdot\sum_{r=1}^{10}{}^{9}C_{r-1}\cdot{}^{10}C_{10-r} = 10\cdot{}^{19}C_9$

Now $\frac{X}{1430} = \frac{10\cdot{}^{19}C_9}{1430} = \frac{{}^{19}C_9}{143} = \frac{{}^{19}C_9}{11\times 13} = \frac{19.17.16}{8} = 19\times 34 = 646$

15. Ans.

The correct option is

(A) P→4; Q→2; R→1; S→1
(B) P→3; Q→3; R→6; S→5
(C) P→4; Q→2; R→1; S→6
(D) P→4; Q→3; R→6; S→5

Ans. (A)

Sol. $E_1 : \frac{x}{x-1} > 0 \Rightarrow x \in (-\infty, 0) \cup (1, \infty)$

$$E_2 : -1 \le \ell n\left(\frac{x}{x-1}\right) \le 1 \Rightarrow \frac{1}{e} \le \frac{x}{x-1} \le e \Rightarrow \frac{1}{e} \le + \frac{1}{x-1} \le e$$

$$\frac{1}{e} - 1 \le \frac{1}{x-1} \le e - 1 \quad \Rightarrow \quad (x-1) \in \left(-\infty, \frac{e}{1-e}\right] \cup \left[\frac{1}{e-1}, \infty\right)$$

$$x \in \left(-\infty, \frac{1}{e-1}\right] \cup \left[\frac{e}{e-1}, \infty\right)$$

Now $\frac{x}{x-1} \in (0, \infty) - \{1\} \ \forall x \in E_1 \quad \Rightarrow \quad \ell n\left(\frac{x}{x-1}\right) \in (-\infty, \infty) - \{0\}$

$$\sin^{-1}\left(\ell n\left(\frac{x}{x-1}\right)\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

16. Ans.

The correct option is

(A) P→4; Q→6; R→2; S→1
(B) P→1; Q→4; R→2; S→3
(C) P→4; Q→6; R→5; S→2
(D) P→4; Q→2; R→3; S→1

Ans. (C)

Sol. 6 Boys & 5 girls
$\alpha_1 \rightarrow$ number of ways of selecting 3 boys & 2 girls ${}^6C_3 \times {}^5C_2 = 200$

$\alpha_2 \to$ Boys & girls are equal & members □ 2

$^6C_1.^5C_1 + ^6C_2.^5C_2 + ^6C_3.^5C_3 + ^6C_4.^5C_4 + ^6C_5.^5C_5 = ^{11}C_5 - 1 = 461$

$\alpha_3 \to$ number of ways of selecting 5 having at least 2 girls

$^{11}C_5 - ^6C_5 - ^6C_4.^5C_1 = ^{11}C_5 - 81 = 381$

$\alpha_4 \to G_1$ is included □ $^4C_1.^5C_2 + ^4C_2.^5C_1 + ^4C_3 = 40 + 30 + 4 = 74$

M_1 is included $\to ^4C_2.^5C_1 + ^4C_3 = 34$

G_1 & M_1 both are excluded $\to ^4C_4 + ^4C_3.^5C_1 + ^4C_2.^5C_2 = 81$

$Total = 74 + 34 + 81 = 189$

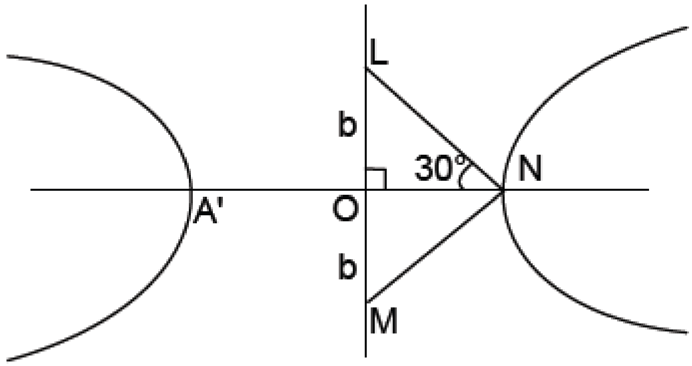
17. Ans.

The correct option is:

(A) P ⟶ 4; Q ⟶ 2; R ⟶ 1; S ⟶ 3
(B) P ⟶ 4; Q ⟶ 3; R ⟶ 1; S ⟶ 2
(C) P ⟶ 4; Q ⟶ 1; R ⟶ 3; S ⟶ 2
(D) P ⟶ 3; Q ⟶ 4; R ⟶ 2; S ⟶ 1

Ans. (B)

Sol.



$\text{Area of } LMN = 4\sqrt{3}$

$\frac{1}{2}(2b)\left(\sqrt{3}b\right) = 4\sqrt{3} \quad \Rightarrow b^2 = 4 \quad b = 2 \quad 2b = 4$

Here $\frac{a}{b} = \cot 30° \Rightarrow a = \sqrt{3}b \Rightarrow a = 2\sqrt{3}$

$b^2 = a^2(e^2 - 1)$

$4 = 12(e^2 - 1)$

$e^2 = 1 + \frac{1}{3} = \frac{4}{3} \Rightarrow e = \frac{2}{\sqrt{3}}$ and $2ae = 2 \times 2\sqrt{3} \times \frac{2}{\sqrt{3}} = 8$

And length of latus rectum = $\frac{2b^2}{a} = \frac{2 \times 4}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$

18. **Ans.**

The correct option is:

(A) P → 2; Q → 3; R → 1; S → 4
(B) P → 4; Q → 1; R → 2; S → 3
(C) P → 4; Q → 2; R → 1; S → 3
(D) P → 2; Q → 1; R → 4; S → 3

Ans. (D)

Sol. (i)

$$f_1'(0) = \lim_{h \to 0} \frac{\sin\sqrt{1 - e^{-h^2}} - 0}{h} = \lim_{h \to 0} \frac{\sin\sqrt{1 - e^{-h^2}}}{\sqrt{1 - e^{-h^2}}} \times \sqrt{\frac{1 - e^{-h^2}}{h^2}} \times \frac{|h|}{h}$$

$$= 1 \times 1 \times \frac{|h|}{h} = 1 \times 1 \times \frac{|h|}{h}$$

= limit does not exist.

$\Rightarrow$ for option (P), (2) is correct.

(ii)

$$\lim_{x \to 0} f_2(x) = \lim_{x \to 0} \frac{|\sin x|}{\tan^{-1} x}$$

$$= \lim_{x \to 0} \frac{|\sin x|}{|x|} \times \frac{x}{\tan^{-1} x} \times \frac{|x|}{x}$$

$$= \lim_{x \to 0} 1 \times 1 \times \frac{|x|}{x}$$

= limit does not exist $\Rightarrow$ for option Q, (1) is correct.

(iii) $\lim_{x \to 0} f_3(x) = \lim_{x \to 0}\left[\sin\left(\log_e(x+2)\right)\right]$

now at x tends to zero $(x+2)$ tends to 2

$\Rightarrow \log_e(x+2)$ tends to $\ell n 2$

which is less than 1

$$0 < \lim_{x \to 0} \sin\left(\log_e(x+2)\right) < \sin 1 \Rightarrow \lim_{x \to 0}\left[\sin\left(\log_e(x+2)\right)\right] = 0$$

$$f_3(x) = \{0 \quad x \in \left[-1, e^{\pi/2} - 2\right)$$

$$\Rightarrow \quad f_3'(x) = 0 \quad \forall x \in \left(-1, e^{\pi/2} - 2\right)$$

$$\Rightarrow \quad f_3''(x) = 0 \quad \forall x \in \left(-1, e^{\pi/2} - 2\right)$$

Hence for (R), (4) is correct.

(iv)

$$\lim_{x \to 0} f_4(x) = \lim_{x \to 0}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x \to 0} x^2\left(\sin\frac{1}{x}\right) = 0$$

$$f_4'(0) = \lim_{h \to 0}\frac{h^2 \sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{h \to 0} h \sin\left(\frac{1}{x}\right) = 0$$

$$f_4''(0) = \frac{-\cos\frac{1}{h} + h\sin\frac{1}{x} - 0}{h} \Rightarrow \text{does not exist}$$

hence for (S), (3) is correct.