JEE MAIN-2000

MATHEMATICS

[Time: 2 hours] [Maximum Marks: 100]

A. General Instructions :

1. There are ten questions in this paper. Attempt all Questions.

2. Answer each question starting on a new page. The corresponding question number must be written in the left margin. Answer all the parts of a question at one place only.

3. Use only Arabic numerals (0, 1, 2 …………9) in answering the questions irrespective of the language in which your answer.

4. Use of logarithmic tables is not permitted.

5. Use of calculator is not permitted.

1. (a) The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that resulting sum is the square of an integer.

(b) For any positive integers m, n $\left(\text{with}\, n \geq m\right)$, let $\binom{n}{m} = {}^{n}C_{m}$ Prove that

$$\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\ldots+\binom{m}{m}=\binom{n+2}{m+2}$$

Hence or otherwise, prove that

$$\binom{n}{m}+2\binom{n-1}{m}+3\binom{n-2}{m}+\ldots+(n-3+1)\binom{m}{m}\binom{n+2}{m+2} \quad (6)$$

2. (a) If α, β , are the roots of $ax2+bx+c=0,(a\neq 0)$ and $\alpha+d, \beta+d$ are the roots of $Ax2+Bx+C=0,(A\neq 0)$ for some constant d , then prove that

$(b2-4ac)/a2=(B2-4AC)//A2$ (4)

b) For every positive integer, prove that

$\sqrt{}(4n+1)<\sqrt{}n+\sqrt{}(n+1)<\sqrt{}(4n+2)$

Hence or otherwise, prove that $[\sqrt{}n+\sqrt{}(n+1)=[\sqrt{}(4n+1]])$, where $[x]$ denotes the greatest integer not exceeding x .

3. (a) In any triangle ABC , prove that

$\cot A/b+\cot B/2+\cot C/2=A/2\cot B/2\cot C/2$ (3)

(b) Let ABC be a triangle with incentre I and inradius r . Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If $r1$, $r2$ and $r3$ are the radii of circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$ respectively, prove that

$r/1(r-r1)+r2/(r-r2)+r3/(r-r3)=(r1\quad r2\quad r3)/(r-r1)(r-r2)(r-r3)$

4. For points $P=(x1, y1)$ and $Q=(x2, y2)$ of the coordinate plane, a new distance $d(P,\ Q)$ is defined by $(P,\ Q)=|x1-x2|+|y1-y2|$. Let $O=(0,\ 0)$ and $A=(3,2)$. Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of line segment of finite length and an infinite ray. Sketch this set in a labelled diagram.

5. Prove that for all values of θ

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0.$$

(b) Let ABC be an equilateral triangle inscribed in the circle $x2+y2=a2$. Suppose perpendiculars from A,B,C to the major axis of the ellipse $x2/a2+y2/b2=1,(a>b)$ meets the ellipse respectively at P,Q,R so that P,Q,R lie on the same side of the major axis as A,B,C respectively. Prove that the normals to the ellipse drawn at the points P,Q and R are concurrent.

6. Let $C1$ and $C2$ be, respectively, the parabolas $x2=y-1$ and $y2=a2$. Let P be any point on $C1$ and Q be any point on $C2$. Let $P1$ and $Q1$ be the reflections of P and Q, respectively, with respect to the line $y=x$. Prove that $P1$ lies on $C2, Q1$, lies on $C1$ and $PQ \geq \min\{PP1, QQ1\}$. Hence or otherwise, determine points $P0$ and $Q0$ on the parabolas $C1$ and $C2$ respectively such that $P0Q0 \leq P0$ for all pairs of points (P, Q) with P on $C1$ and Q on $C2$.

7. (a) Suppose $P(x)=a_c+a_1x+a_2x^2+\ldots+a_nx^n$. If $|p(x)| \leq |e^{x-1}-1|$ for all $x \geq 0$, prove that $|a_1+2a_2+\ldots+na_n| \leq 1$

(b) For $x>0$, let $f(x)=\int_1^x \frac{\ln t}{1+t}dt$. Find the function $f(x)+f(1/x)$ and show that $f(e)+f(1/e)=1/2$. Here In $t=\log e^t$.

8. A country has food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than or equal to $(\text{In}10-\text{In}9)/(\text{In}(1.04)-0.03)$.

9. (a) A coin has probability p of showing head when tossed. It is tossed n times. Let pn, denote the probability that no two (or more) consecutive heads occur. Prove that $p1=1, p2=1-p2$ and $np=(1-p).pn-1+p(1-p)\,pn-2$ for all $n\geq 3$.

(b) In (a), prove by induction on n, that $pn=A\alpha n+B\beta n$ for all $n\geq 1$, where α and β are the roots of the quadratic $x2-(1-p)x-p(1-p)=0$ and

$$A=\frac{p^2+\beta-1}{\alpha\beta-\alpha^2}, B=\frac{p^2+\alpha-1}{\alpha\beta-\beta^2}.$$

10. Let ABC and PQR be any two triangles in the same plane. Assume that the perpendiculars from the points A, B, C to the sides QR, RP, PQ respectively are concurrent. Using vector methods or otherwise, prove that the perpendiculars from P, Q, R to BC, CA, AB respectively are also concurrent.