IIT-JEE-2001

MATHEMATICS

MAINS

1. Let $a1$, $a2$, …… be positive real numbers in geometric progression. For each n let An, Gn, Hn be respectively, the arithmetic mean, geometric mean, and harmonic mean of $a1$, $a2$, ……, an. Find an expression for the geometric mean of $G1, G2, \ldots\ldots, Gn$ in terms of $A1, A2, \ldots\ldots, An, H1, H2, \ldots\ldots, Hn$.

2. Let a, b, c be positive real numbers such that $b2-4ac>0$ and let $a1=c$. Prove by induction that

$$an+1=(a\alpha n2)/((b2-2a(\alpha 1-\alpha 2+\ldots\ldots+\alpha n))$$

is well-defined and $an+1<\alpha\_n/2$ for all $n=1,2\ldots\ldots$

(Here, ‘well-defined’ means that the denominator in the expression for $an+1$ is not zero.)

3. Let $-1<p<1$. Show that the equation $4x2-3x-p=0$ has a unique root in the interval $[1/2,1]$ and identify it.

4. Let $2x2+y2-3xy=0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA.

5. Evaluate $\int \sin-1((2x+2)/\sqrt{}(4x+2+8x+13))dx$.

6. Let $f(x), x>0$, be a non-negative continuous function, and let $F(x)=\int 0x\, f(t)dt, x>0$. If for some $c>0, f(x)<cF(x)$ for all $x>0$, then show that $f(x)=0$ for all $x>0$.

7. Let $b^{1}0$ and for $j = 0,1,2,\ldots\ldots,n$, let Sj be the area of the region bounded by the y-axis and the curve $xeay = \sin$ by, $\pi/b \le y \le ((j+1)\pi)/b$.

Show that $S0,\ S1,\ S2,\ldots\ldots,Sn$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$.

8. Let $\alpha\,\hat{I}\,R$. Prove that a function $f : R\text{-->}R$ is differentiable at α if and only if there is a function $g : R\text{-->}R$ which is continuous at α and satisfies $f(x) - f(\alpha) = f(x)(x-\alpha)$ for all $x\,\hat{I}\,R$.

9. Let $C1$ and $C2$ be two circles with $C2$ lying inside $C1$. A circle C lying inside $C1$ touches $C1$ internally and $C2$ externally. Identify the locus of the centre of C.

10. Show, by vector methods, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices.

11. (a) $a,\ b,\ c$ be real numbers with $a^{1}0$ and let $a,\ b$ be the roots of the equation $ax2 + bx + c = 0$. Express the roots of $a2x2 + abcx + c3 = 0$ in terms of a, b.

(b) Let $a,\ b,\ c$ real numbers with $a2 + b2 + c2 = 1$. Show that the equation

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx-ay & -ax+by-c & cy+b \\ cx-a & cy+b & -ax-by+c \end{vmatrix} = 0 \text{ represents a straight line.}$$

12. (a) Let P be a point on the ellipse $x2/a2 + y2/b2 = 1, 0 < b < a$. Let the line parallel to y-axis passing through P meet the circle $x2 + y2 = a2$ at the point Q such that P and Q are on the same side of x- axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR : PQ = r : s$ as P varies over the ellipse.

(b) If D is the area of a triangle with side lengths $a,\ b,\ c$ then show that $D < 1/4\sqrt{((a+b+c)abc)}$.

Also show that the equality occurs in the above inequality if and only if $a = b = c$.

13. A hemispherical tank of radius $2\,\text{metres}$ is initially full of water and has an outlet of $12\,\text{cm}2$ cross sectional are at the bottom. The outlet is opened at some instant.

The flow through the outlet is according to the law $v(t) = 0.6\sqrt{(2gh(t))}$, where $v(t)$ and $h(t)$ are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t ,and g is the acceleration due to gravity. Find the time it takes to empty the tank.

(Hint : Form a differential equation by relating the decrease of water level to the outflow).

14. (a) An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn now is white?

(b) An unbiased die, with faces numbered 1, 2, 3, 4, 5, 6 is thrown n times and the list of n numbers showing up is noted. What is the probability that, among the numbers 1, 2, 3, 4, 5, 6 only three numbers appear in this list?

15. (a) Find 3-dimensional vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ satisfying

$\overrightarrow{v_1}, \overrightarrow{v_1} = 4, \overrightarrow{v_1}, \overrightarrow{v_2} = -2, \overrightarrow{v_1}, \overrightarrow{v_3} = 6, \overrightarrow{v_2}, \overrightarrow{v_2} = 2, \overrightarrow{v_2}, \overrightarrow{v_3} = -5, \overrightarrow{v_3}, \overrightarrow{v_3} = 29$.

(b) Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $B(t) = g_1(t)\hat{i} + g_2(t)\hat{j}, t \in [0,1]$, where f_1, f_2, g_1, g_2 are continuous functions. If $\vec{A}(t)$ and $\vec{B}(t)$ are non-zero vectors for all t and $\vec{A}(0) = 2\hat{i} + 3\hat{j}, \vec{A}(1) = 6\hat{i} + 2\hat{j}, \vec{B}(0) = 3\hat{i} + 2\hat{j}$ and $\vec{B}(1) = 2\hat{i} + 6\hat{j}$. The show that $\vec{A}(t)$ and $\vec{B}(t)$ are parallel for some t.