AIEE-2002

MATHEMATICS

6. Sol.

$$I = AR^{p-1} \Rightarrow \log I = \log A + (p-1)\log R$$
$$m = AR^{p-1} \Rightarrow \log m = \log A + (q-1)\log R$$
$$n = AR^{r-1} \Rightarrow \log n = \log A + (q-1)\log R$$

Now,

$$\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (p-1)\log R & q & 1 \\ \log A + (p-1)\log R & r & 1 \end{vmatrix} = 0$$

7. Sol.

$$\operatorname*{Lim}_{x\to 0} \frac{\sqrt{1-\cos 2x}}{\sqrt{2}x} \Rightarrow \operatorname*{Lim}_{x\to 0} \frac{\sqrt{1-(1-2\sin^2 x)}}{\sqrt{2}x}; \operatorname*{Lim}_{x\to 0} \frac{\sqrt{2\sin^2 x}}{\sqrt{2}x} \Rightarrow \operatorname*{Lim}_{x\to 0} \frac{|\sin x|}{x}$$

the function does not exist or $LHS \neq RHS$

8. Sol.

$$AB = \sqrt{(4+1)^2 + (0+1)^2} = \sqrt{26}; BC = \sqrt{(3+1)^2 + (5+1)^2} = \sqrt{52}$$

$CA = \sqrt{(4-3)^2 + (0-5)^2} = \sqrt{26}$; So, in isosceles triangle $AB = CA$

For right angled triangle $BC^2 = AB^2 + AC^2$

So, here $BC = \sqrt{52}$ ro $BC^2 = 52$ or $\left(\sqrt{26}\right)^2 + \left(\sqrt{26}\right)^2 = 52$

So, given triangle is right angled and also isosceles

9. Sol.

Total student $=100; \text{for } 70 stds 75\times 70=5250 \Rightarrow 7200-5250=1950$

Average of girls $=\dfrac{1950}{30}=65$

10. Sol.

$$\cot^{-1}\left(\sqrt{\cos\alpha}\right)-\tan^{-1}\left(\cos\alpha\right)=\alpha$$

$$\tan^{-1}\left(\frac{1}{\sqrt{\cos\alpha}}\right)-\tan^{-1}\left(\sqrt{\cos\alpha}\right)=x \Rightarrow \tan^{-1}\frac{\frac{1}{\sqrt{\cos\alpha}}-\sqrt{\cos\alpha}}{1+\frac{1}{\sqrt{\cos\alpha}}.\sqrt{\cos\alpha}}$$

$$\Rightarrow \tan^{-1}\frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}=x \Rightarrow \tan x=\frac{1-\cos\alpha}{2\sqrt{\cos\alpha}} \text{ OR } \cot x=\frac{2\sqrt{\cos\alpha}}{1-\cos\alpha} or \cos ec\, x=\frac{1+\cos\alpha}{1-\cos\alpha}$$

$$\sin x=\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{1-\left(1-2\sin^2\alpha/2\right)}{1+2\cos^2\alpha/2-1} \text{ or } \sin x=\tan^2\frac{\alpha}{2}$$

11. Sol.

Order $=3$, degree $=3$

12. Sol.

$$\frac{x-4}{1}=\frac{y-7}{5}=\frac{z-4}{4} \quad \text{K K (i)}$$

$$a(x-4)+b(y-7)+c(z-4)=0 \quad \text{K K (ii)}$$

Line passing through point $(3,2,0)$

$$a+5x+4x+0 \quad \text{K K (iii)}$$

Solving the equation we get by equation (ii)

$$x-y+z=1$$

13. Sol.

$$\frac{d^2y}{dx^2}=e^{-2x};\frac{dy}{dx}=\frac{e^{-2x}}{-2}+c;y=\frac{e^{-2x}}{4}+cx+d$$

14. Sol.

$$\underset{x->\infty}{\text{Lim}}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^{\frac{1}{x}}=\underset{x\to\infty}{\text{Lim}}\left(\frac{1+\frac{5}{x}+\frac{3}{x^2}}{1+\frac{1}{x}+\frac{3}{x^2}}\right)^{\frac{1}{x}}=1$$

15. Sol.

$$f(x)=\sin^{-1}\left(\log_3\left(\frac{x}{3}\right)\right)\text{ exists if}$$

$$-1\le\log_3\left(\frac{x}{3}\right)\le 1\Leftrightarrow 3^{-1}\le\frac{x}{3}\le 3^1\Leftrightarrow 1\le x\le 9\text{ or }x\in[1,9]$$

17. Sol.

$$ar^4=2$$
$$a\times ar\times ar^2\times ar^3\times ar^4\times ar^5ar^6\times ar^7\times ar^8$$
$$=a^9r^{36}\left(ar^4\right)=2^9=512$$

18. Sol.

$$\int_0^{10\pi} |\sin x| dx = 10\left[\int_0^{\pi/2} \sin x\, dx + \int_{\pi/2}^{\pi} \sin x\, dx\right]$$

$$= 10 \times [\cos]_0^{\pi/2} + [\cos x]_{\pi/2}^{\pi}; \quad 10[1+1] = 10\times = 20$$

19. Sol.

$$\int_0^{\pi/4} \tan^n x\left(1+\tan^2 x\right) dx = \int_0^{\pi/4} \tan^2 x \sec^2 x dx = \int_0^1 t^2 dt \text{ where } t = \tan x$$

$$l_n + l_{n+2} = \frac{1}{n+1}; \Rightarrow \lim_{x\to\Delta} n\left[l_n + l_{n+2}\right] n.\frac{1}{n+1} = \frac{n}{n+1} = \frac{n}{n\left(1+\frac{1}{n}\right)} = 1$$

20. Sol.

$$\int_1^0 \left[x^2\right] dx + \int_1^{\sqrt{2}} \left[x^2\right] dx = 0 + \int_1^{\sqrt{2}} dx = \sqrt{2} - 1$$

21. Sol.

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} + 2\int_{-\pi}^{\pi} \frac{x \sin x}{1+\cos^2 x}$$

$$= 0 + 4\int_{-\pi}^{\pi} \frac{x \sin x dx}{1+\cos^2 x} l = 4\int_{-\pi}^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}$$

$$l = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} \Rightarrow l = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} - 4\pi\int \frac{x \sin x}{1+\cos^2 x} \Rightarrow 2l = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

put $\cos x = t$ and solve it.

22. Sol.

We have, $\lim\limits_{x\to 2}\dfrac{xf(2)-2f(x)}{x-2}\left(\dfrac{0}{0}\right)=\lim\limits_{x\to 2} f(2)-2f'(x)=f(2)-2f'(2)=4-2\times 4=-4$

23. Sol.

Let $|z|=|\omega|=r \quad \therefore z=re^{i\theta}, \omega=re^{i\varphi}$ where $\theta+\varphi=\pi \quad \therefore \bar{\omega}=re^{-i\varphi}$

$\therefore z=re^{i(\pi-\varphi)}=re^{i\pi}.e^{-i\varphi}=-re^{-i\varphi}=-\bar{\omega}$

24. Sol.

Given $|z-4|<|z-2|$ Let $z=x+iy$

$\Rightarrow |(x-4)+iy|<|(x-2)+iy| \Rightarrow (x-4)^2+y^2<(x-2)^2+y^2$

$\Rightarrow x^2-8x+16<x^2-4x+4 \Rightarrow 12<4x \Rightarrow x>3 \Rightarrow \text{Re}(z)>3$

26. Sol.

Let a = first term of $G.P.$

$r=$ common ratio of $G.P.$; Then $G.P.$ is a, ar, ar^2

Given $s_\infty=20 \Rightarrow \dfrac{a}{1-r}=20 \Rightarrow a=20(1-r) \quad \text{K K (i)}$

Also $a^2+a^2r^2+a^2r^4+\text{K to}\infty=100 \Rightarrow \dfrac{a^2}{1-r^2}=100 \Rightarrow a^2=100(1-r)(1+r)\text{K K (ii)}$

From (i), $a^2=400(1-r)^2$; From (ii) and (iii), we get

$100(1-r)(1+r)=400(1-r)^2 \quad \text{K K (iii)}$

$\Rightarrow 1+r=4-4r \Rightarrow 5r=3 \Rightarrow r=3/5$

27. Sol.

$$1^3 - 2^3 + 3^3 - 4^3 + K\,K + 9^3$$
$$= 1^3 + 3^3 + 5^3 + K + 9^3 - \left(2^3 + 4^3 + K + 8^3\right)$$
$$= s_1 - s_2$$
$$\text{For } s_1, t_n = (2n-1)^3 = 8n^3 - 12n^2 + 6n - 1$$
$$s_1 = \sum t_n = 8\sum n^3 - 12\sum n^2 + 6\sum n - \sum_1$$
$$= \frac{8n^2(n+1)^2}{4} - \frac{12n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} - n$$

Here $n = 5$. Hence $S_1 = 2\times25\times36 - 2\times5\times6\times11 + 3\times30 - 5$
$= 1800 - 660 + 90 - 5 = 1890 - 665 = 1225$

For $S_2, t_n = 8n^3; S_2 = \sum t_n = 8\sum n^3 = \dfrac{8n^2(n+1)^2}{4} = 2\times16\times25 = 800.(\text{for } n = 4)$

$\therefore$ Required sum $= 1225 - 800 = 425$

28. Sol.

Let α, β and y, δ are the roots of the equations

$x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $\therefore \alpha + \beta = -a, \alpha\beta = b$ and $y + \delta = -b, y\delta = a$

Given $\alpha - \beta = y - \delta \Rightarrow (\alpha-\beta)^2 = (y-\delta)^2 \Rightarrow (\alpha+\beta)^2 - 4\alpha\beta = (y+\delta)^2\,4y\delta$

$\Rightarrow a^2 - 4b = b^2 - 4a \Rightarrow \left(a^2 - b^2\right) + 4(a-b) = 0 \Rightarrow a + b + 4 = 0\,(Q\, a \neq b)$

30. Sol.

$p+q=-p$ and $pq=q \Rightarrow (p-1)=0 \Rightarrow q=0$ or $p=1$

If $q=0$, then $p=0$. i.e. $p=q \quad \therefore p=1 \text{and} q=-2$

31. Sol.

$$ab+bc+ca=\frac{(a+b+c)^2-1}{2}<1$$

32. Sol.

Required number of numbers $=5\times6\times6\times4=36\times20=720$

33. Sol.

Required number of numbers $=3\times5\times5\times5=375$

34. Sol.

Required numbers are $5!+5!4!=216$

35. Sol.

Required

$\text{sum} = (1+4+6+\text{K} +100)+(5+10+15+\text{K} +100)-(10+20+\text{K} +100) = 2550+1050-530 = 3050$

36. Sol.

We have $t_{p+1} = {}^{p+q}C_p x^p$ and $t_{q+1} + {}^{p+q}C_q x^q \quad {}^{p+q}C_p = {}^{p+q}C_q$.

37. Sol.

We have $2^n = 4096 = 2^{12} \Rightarrow n = 12;$ So middle term $= t_7; t_7 = t_{6+1} = {}^{12}C_6 = \frac{12!}{6!6!} = 934$

39. Sol.

$t_{r+2} = {}^{2n}C_{r+1} x^{r+1}; t_{3r} = {}^{2n}C_{3r-1} x^{3r-1}$

Given ${}^{2n}C_{r+1} = {}^{2n}C_{3r-1} \Rightarrow {}^{2n}C_{2n(r+1)} = {}^{2n}C_{3r-1} \Rightarrow 2n-1 = 3r-1 \Rightarrow 2n = 4r$