IIT-JEE-2003

MATHEMATICS

MAINS

1. If z_1 and z_2 are two complex numbers such that

$|z_1| < 1 < |z_1|$ then prove that

$$\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1.$$

2. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum.

3. If matrix

$$A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

where $a,\ b,\ c$ are real positive numbers, $abc = 1$ and $A^T\ A = I$, then find the value of $a^3 + b^3 + c^3$.

4. Prove that

$$2^k \binom{n}{0}\binom{n}{k} - 2^{n-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} - \ldots (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}.$$

5. If f is an even function then prove that

$$\int_0^{\pi/2} f(\cos 2x)\cos x\,dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\,dx.$$

6. For a student to qualify, he must pass at least two out of three exams. The probability that he will pass the 1st exam is P. If he fails in one of the exams then the probability of his passing in the next exam is $P/2$ otherwise it remains the same. Find the probability that he will qualify.

7. For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6,8)$ to the circle and the chord of contact is maximum.

8. Prove that there exists no complex number z such that $|z| < 1/3$ and

$\sum r = 1^n a_r z^r = 1$ where $_r\|a < 2$.

9. A is targeting to B, B and C targeting to A. Probability of hitting the target by A, B and C are $2/3, 1/2$ and $1/3$ respectively. If A is hit then find the probability that B hits the target and C does not.

10. If a function $f:[-2a,2a]-->R$ is an odd function such that $f(x) = f(2a-x)$ for $x\hat{I}[a,2a[$ and the left hand derivative at $x = a$ is 0 then find the left hand derivative at $x = -a$.

11. Using the relation $2(1-\cos x) < x^2, x^1 0$ or otherwise, prove that $\sin(\tan x) > x \forall x\hat{I}[0\pi/4]$.

12. If a,b,c are in $A.P.,a^2,b^2,c^2$ are in $H.P.$, then prove that either $a=b=c$ or $a,b,-c/2$ form a $G.P.$

13. If $x^2+(a-b)x+(1-a-b)=0$ where $a,b\hat{I}R$ then find the values of a for which equation has unequal real roots for all values of b.

14. Normals are drawn from the point P with slopes m_1,m_2,m_3 to the parabola $y^2=4x$. If locus of P with $m_1\ m_2=a$ is a part of the parabola itself then finda.

15. If the function $f:[0,4]-->R$ is differentiable then show that

(i) For $a,b\hat{I}(0,4),(f(4))^2=f'(a)f(b)$

(ii) $\int_0^4 f(t)dt=2\left[\alpha f(\alpha^2)+\beta f(\beta)^2)\right]\forall 0<\alpha,\beta<2$.

16. (i) Find the equation of the plane passing through the points $(2,1,0),(5,0,1)$ and $(4,1,1)$

(ii) If P is the point $(2,1,6)$ then the point Q such that PQ is perpendicular to the plane in (i) and the mid point of PQ lies on it.

17. If $P(1)=0$ and $(dp(x))/dx>P(x)$ for all $x>1$ then prove that $P(x)>0$ for all $x>1$.

18. If In is the area of n sided regular polygon inscribed in a circle of unit radius and On be the area of the polygon circumscribing the given circle, prove that

$$I_n=\frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$$

19. If $\vec{u}$, $\vec{v}$, $\vec{w}$ are three non-coplanar unit vectors and α, β, γ are the angles between $\vec{u}$ are and $\vec{v}$, $\vec{v}$ are and $\vec{w}$, $\vec{w}$ and $\vec{u}$ are respectively and $\vec{x}, \vec{y}, \vec{z}$ are unit vectors along the bisectors of the angles a, b, g respectively. Prove that

$$\left[\vec{x}\times\vec{y} \quad \vec{y}\times\vec{z} \quad \vec{z}\times\vec{x}\right]=\frac{1}{16}\left[\vec{u}\,\vec{v}\,\vec{w}\right]^2 \sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$

20. A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant $= k > 0$). Find the time after which the cone is empty.