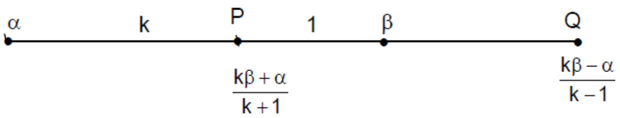
JEE MAIN - 2004

MATHEMATICS

[Time : 2 hours]

Note: Question number 1 to 10 carries **2 marks** each and 11 to 20 carries **4 marks** each.

1. Sol.



Centre is the mid-point of points dividing the join of α and β in the ratio $k:1$ internally and externally.

i.e. $z=\frac{1}{2}\left(\frac{k\beta+\alpha}{k+1}+\frac{k\beta-\alpha}{k-1}\right)=\frac{\alpha-k^2\beta}{1-k^2}$

radius $=\left|\frac{\alpha-k^2\beta}{1-k^2}-\frac{k\beta+\alpha}{1+k}\right|=\left|\frac{k(\alpha-\beta)}{1-k^2}\right|$.

Alternative:

We have $\frac{|z-\alpha|}{|z-\beta|}=k$

so that $(z-\alpha)(\bar{z}-\bar{\alpha})=k^2(z-\beta)(\bar{z}-\bar{\beta})$

or $z\bar{z}-\alpha\bar{z}-\bar{\alpha}z+\alpha\bar{\alpha}=k^2\left(z\bar{z}-\beta\bar{z}-\bar{\beta}z+\beta\bar{\beta}\right)$

or $z\bar{z}(1-k^2)-(\alpha-\kappa^2\beta)\bar{z}-(\bar{\alpha}-\kappa^2\bar{\beta})z+\alpha\bar{\alpha}-k^2\beta\bar{\beta}=0$

or $z\bar{z}-\frac{(\alpha-k^2\beta)}{1-k^2}\bar{z}-\frac{(\bar{\alpha}-k^2\bar{\beta})}{1-k^2}z+\frac{\alpha\bar{\alpha}-k^2\beta\bar{\beta}}{1-k^2}=0$

which represents a circle with centre $\frac{\alpha-k^2\beta}{1-k^2}$ and radius

$\sqrt{\frac{(\alpha-k^2\beta)(\bar{\alpha}-k^2\bar{\beta})}{(1-k^2)^2}-\frac{\alpha\bar{\alpha}-k^2\beta\bar{\beta}}{(1-k^2)}}=\left|\frac{k(\alpha-\beta)}{1-k^2}\right|$.

2. Sol.

Given that $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$

$\Rightarrow \vec{a}\times\left(\vec{b}-\vec{c}\right)=\left(\vec{c}-\vec{b}\right)\times\vec{d}=\vec{d}\times\left(\vec{b}-\vec{c}\right)\Rightarrow \vec{a}-\vec{d}\parallel \vec{b}-\vec{c}$

$\Rightarrow\left(\vec{a}-\vec{d}\right)\cdot\left(\vec{b}-\vec{c}\right)\neq 0\Rightarrow \vec{a}\cdot\vec{b}+\vec{d}\cdot\vec{c}\neq\vec{d}\cdot\vec{b}+\vec{a}\cdot\vec{c}.$

3. Sol.

Let there be n^2 objects distributed in n groups, each group containing n identical objects. So number of arrangement of these n^2 objects are $\frac{n^2!}{(n!)^n}$ and number of arrangements has to be an integer.

Hence $\frac{n^2}{(n!)^n}$ is an integer.

4. Sol.

$(M-I)^T=M^T-I=M^T-M^TM=M^T\left(I-M\right)$

$\Rightarrow\left|(M-I)^T\right|=|M-I|=\left|M^T\right|\left|I-M\right|=\left|I-M\right|\Rightarrow|M-I|=0.$

Alternate: $\det(M-I)=\det(M-I)\det\left(M^T\right)=\det\left(MM^T-M^T\right)$

$=\det\left(I-M^T\right)=-\det\left(M^T-I\right)=-\det(M-I)^T=-\det(M-I)\Rightarrow\det(M-I)=0.$

5. Sol.

$$y=\int_{\pi^2/16}^{x^2}\frac{\cos x.\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta=\cos x\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$$

so that $\frac{dy}{dx}=-\sin x\int_{\pi^2/16}^{x^2}\frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta+\frac{2x\cos x\cdot\cos x}{1+\sin^2 x}$

Hence, at $x=\pi,\frac{dy}{dx}=0+\frac{2\pi(-1)(-1)}{1+0}=2\pi.$

6. Sol.

Let the equation of the plane $ABCD$ be $ax+by+cz+d=0$, the point A'' be (α,β,γ) and the height of the parallelopiped $ABCD$ be h.

$\Rightarrow\frac{|a\alpha+b\beta+c\gamma+d|}{\sqrt{a^2+b^2+c^2}}=0.9\text{h}.\Rightarrow a\alpha+b\beta+c\gamma+d=\pm0.9h\sqrt{a^2+b^2+c^2}$

$\Rightarrow$ the locus of A'' is a plane parallel to the plane $ABCD$.

7. Sol.

$$\lim_{n\to\infty}\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n=\lim_{n\to\infty}n\left[\frac{2}{\pi}\left(1+\frac{1}{n}\right)\cos^{-1}\frac{1}{n}-1\right]$$

$=\lim_{n\to\infty}nf\left(\frac{1}{n}\right)=f'(0)$ where $f(x)=\frac{2}{\pi}(1+x)\cos^{-1}x-1.$

Clearly, $f(0)=0$.

Now, $f'(x)=\frac{2}{\pi}\left[(1+x)\frac{-1}{\sqrt{1-x^2}}+\cos^{-1}x\right]$

$$\Rightarrow f'(0)=\frac{2}{\pi}\left[-1+\frac{\pi}{2}\right]=\frac{2}{\pi}\left[\frac{\pi-2}{2}\right]=1-\frac{2}{\pi}.$$

8. Sol.

Let $g(x)=\int p(x)dx=\frac{51x^{102}}{102}-\frac{2323x^{101}}{101}-\frac{45x^2}{2}+1035x+c$

$=\frac{1}{2}x^{102}-23x^{101}-\frac{45}{2}x^2+1035x+c.$

Now $g\left(45^{1/100}\right)=\frac{1}{2}(45)^{\frac{102}{100}}-23(45)^{\frac{101}{100}}-\frac{45}{2}(45)^{\frac{2}{100}}+1035(45)^{\frac{1}{100}}+c=c$

$g(46)=\frac{(46)^{102}}{2}-23(46)^{101}-\frac{45}{2}(46)^2+1035(46)+c=c.$

So $g'(x)=p(x)$ will have atleast one root in given interval.

9. Sol.

Let (l,m,n) be the direction ratios of the normal to the required plane so that $l-n=0$ and $-l+m=0$

$\Rightarrow l=m=n$ and hence the equation of the plane containing $(1,1,1)$ $is\,\frac{x}{3}+\frac{y}{3}+\frac{z}{3}=1$.

Its intercepts with the coordinate axes are $A(3,\ 0,\ 0);B\ (0,\ 3,\ 0);\ C(0,\ 0,\ 3)$. Hence the volume of $OABC$

$=\frac{1}{6}\begin{vmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{vmatrix}=\frac{27}{6}=\frac{9}{2}$ cubic units.

10. Sol.

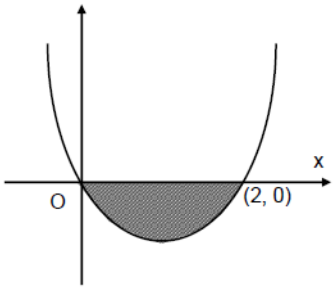
$P(A\cup B).P(A')P(B')\le(P(A)+P(B))P(A')+P(B')$

$=P(A).P(A')P(B')+P(B)P(A')P(B')$

$=P(A)P(B')(1-P(A))+P(B)P(A')(1-P(B))$

$\le P(A)P(B')+P(B)P(A')=P(C).$

11. Sol.



$$\frac{dy}{dx}=\frac{(x+1)^2+y-3}{x+1}$$

or, $\frac{dy}{dx}=(x+1)+\frac{y-3}{x+1}$

Putting $x+1=X, y-3=Y$

$$\frac{dY}{dX}=X+\frac{Y}{X}$$

$$\frac{dY}{dX}-\frac{Y}{X}=X$$

$$I.F.=\frac{1}{X}\Rightarrow\frac{1}{X}\cdot Y=X+c$$

$$\frac{y-3}{x+1}=(x+1)+c.$$

It passes through $(2,0)\Rightarrow c=-4$.

So, $y-3=(x+1)^2-4(x+1)$

$\Rightarrow y=x^2-2x.$

$\Rightarrow$ Required area $=\left|\int_0^2(x^2-2x)dx\right|=\left|\left[\frac{x^3}{3}-x^2\right]_0^2\right|=\frac{4}{3}$sq.units.

12. Sol.

Let the circle with tangent $2x+3y+1=0$ at $(1,-1)$ be

$$(x-1)^2+(y+1)^2+\lambda(2x+3y+1)=0$$

or $x^2+y^2+x(2\lambda-2)+y(3\lambda+2)+2+\lambda=0.$

It is orthogonal to $x(x+2)+(y+1)(y-3)=0$

Or $x^2+y^2+2x-2y-3=0$

so that $\frac{2(2\lambda-2)}{2}\cdot\left(\frac{2}{2}\right)+\frac{2(3\lambda+2)}{2}\left(\frac{-2}{2}\right)=2+\lambda-3\Rightarrow\lambda=-\frac{3}{2}.$

Hence the required circle is $2x^2+2y^2-10x-5y+1=0.$

13. Sol.

Any point on the parabola is $P\left(1+t^2,1+2t\right)$. The equation of the tangent at P is $t(y-1)=x-1+t^2$ which meets the directrix $x=0$ at $Q\left(0,1+t-\frac{1}{t}\right)$. Let R be (h,k).

Since it divides QP externally in the ratio $\frac{1}{2}:1$ Q is the mid point of RP

$$\Rightarrow 0=\frac{h+1+t^2}{2}\text{ or } t^2=-(h+1)$$

and $1+t-\frac{1}{t}=\frac{k+1+2t}{2}$ or $t=\frac{2}{1-k}$

So that $\frac{4}{(1-k)^2}+(h+1)=0$ Or $(k-1)^2(h+1)+4=0$

Hence locus is $(y-1)^2(x+1)+4=0$.

14. Sol.

$$I=\int_{-\pi/3}^{\pi/3}\frac{\left(\pi+4x^{3}\right)dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}$$

$$2I=\int_{-\pi/3}^{\pi/3}\frac{2\pi dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}=\int_{0}^{\pi/3}\frac{2\pi dx}{2-\cos\left(x+\frac{\pi}{3}\right)}$$

$$I=\int_{\pi/3}^{2\pi/3}\frac{2\pi dt}{2-\text{cost}}\Rightarrow I=2\pi\int_{\pi/3}^{2\pi/3}\frac{\sec^{2}\frac{t}{2}dt}{1+3\tan^{2}\frac{t}{2}}=2\pi\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{2dt}{1+3\text{t}^{2}}=\frac{4\pi}{3}\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{dt}{\left(\frac{1}{\sqrt{3}}\right)^{2}\text{t}^{2}}$$

$$I=\frac{4\pi}{3}\sqrt{3}\left[\tan^{-1}\sqrt{3}t\right]_{1/\sqrt{3}}^{\sqrt{3}}=\frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3-\frac{\pi}{4}\right]=\frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right).$$

15. Sol.

$$(1+a)(1+b)(1+c)=1+ab+a+b+c+abc+ac+bc$$

$$\Rightarrow\frac{(1+a)(1+b)(1+c)-1}{7}\geq(ab.a.b.c.abc.ac.bc)^{1/7}\ (\text{using } AM\geq GM)$$

$$\Rightarrow(1+a)(1+b)(1+c)-1>7\left(a^{4}.b^{4}.c^{4}\right)^{1/7}$$

$$\Rightarrow(1+a)(1+b)(1+c)>7\left(a^{4}.b^{4}.c^{4}\right)^{1/7}$$

$$\Rightarrow(1+a)^{7}(1+b)^{7}(1+c)^{7}>7^{7}\left(a^{4}.b^{4}.c^{4}\right).$$

16. Sol.

$$f\left(0^{+}\right)=f\left(0^{-}\right)=f\left(0\right)$$

Here $f\left(0^{+}\right)=\lim_{x\to\infty}\frac{e^{\frac{ax}{2}}-1}{x}=\lim_{x\to\infty}\frac{e^{\frac{ax}{2}}-1}{\frac{ax}{2}}.\frac{a}{2}=\frac{a}{2}.$

$$\Rightarrow b\sin^{-1}\frac{c}{2}=\frac{a}{2}=\frac{1}{2}\Rightarrow a=1.$$

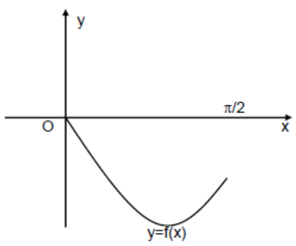
$$Lf'\left(0_{-}\right)=\lim_{h\to 0^{-}}\frac{b\sin^{-1}\frac{\left(h+c\right)}{2}-\frac{1}{2}}{h}=\frac{b/2}{\sqrt{1-\frac{c^{2}}{4}}}$$

$$Rf'\left(0_{+}\right)=\lim_{h\to 0^{+}}\frac{\frac{e^{h/2}-1}{h}-\frac{1}{2}}{h}=\frac{1}{8}$$

Now $Lf'\left(0_{-}\right)=Rf'\left(0_{+}\right)\Rightarrow\frac{\frac{b}{2}}{\sqrt{1-\frac{c^{2}}{4}}}=\frac{1}{8}$

$$4b=\sqrt{1-\frac{c^{2}}{4}}\Rightarrow 16b^{2}=\frac{4-c^{2}}{4}\Rightarrow 64b^{2}=4-c^{2}.$$

17. Sol.



Let $f\left(x\right)=3x^{2}+\left(3-2\pi\right)x-\pi\sin x$

$$f\left(0\right)=0,\ f\left(\frac{\pi}{2}\right)=-ve$$

$f'(x)=6x+3-2\pi-\pi\cos x$

$f''(x)=6+\pi\sin x>0$

$\Rightarrow f'(x)$ is increasing function in $\left[0,\frac{\pi}{2}\right]$

$\Rightarrow$ there is no local maxima of $f(x)$ in $\left[0,\frac{\pi}{2}\right]$

$\Rightarrow$ graph of $f(x)$ always lies below the x-axis in $\left[0,\frac{\pi}{2}\right]$

$\Rightarrow \text{f}(x)\le 0 \text{ in} x\in\left[0,\frac{\pi}{2}\right]$.

$3x^2+3x\le 2\pi x+\pi\sin x \Rightarrow \sin x+2x\ge\frac{3x(x+1)}{\pi}$.

18. Sol.

$AX=U$ has infinite solutions $\Rightarrow |A|=0$

$$\begin{vmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{vmatrix}=0\Rightarrow ab=1 \text{ or } c=d$$

and $|A_1|\begin{vmatrix} a & 0 & f \\ 1 & c & g \\ 1 & d & h \end{vmatrix}=0\Rightarrow g=h; |A_2|\begin{vmatrix} a & f & 1 \\ 1 & g & b \\ 1 & h & b \end{vmatrix}=0\Rightarrow g=h$

$$|A_3|\begin{vmatrix} f & 0 & 1 \\ g & c & b \\ h & d & b \end{vmatrix}=0\Rightarrow g=h, c=d\Rightarrow c=d \text{ and } g=h$$

$BX=V$

$$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0 \quad (\text{since } C_2 \text{ and } C_3 \text{ are equal}) \quad \Rightarrow BX = V \text{ has no unique solution.}$$

$$\text{and } |B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0 (\text{since } c = d, g = h)$$

$$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df \ (\text{since } c = d)$$

$$|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} = a^2df$$

since if $adf \neq 0$ then $|B_2| = |B_3| \neq 0$. Hence no solution exist.

19. Sol.

Let $P(A)$ be the probability that atleast 4 white balls have been drawn.

$P(A_1)$ be the probability that exactly 4 white balls have been drawn.

$P(A_2)$ be the probability that exactly 5 white balls have been drawn.

$P(A_3)$ be the probability that exactly 6 white balls have been drawn.

$P(B)$ be the probability that exactly 1 white ball is drawn from two draws.

$$P(B/A) \frac{\sum_{i=1}^{3} P(A_i) P(B/A_i)}{\sum_{i=1}^{3} P(A_i)} = \frac{\frac{^{12}C_2\, ^6C_4}{^{18}C_6} \cdot \frac{^{10}C_1\, ^2C_1}{^{18}C_6} + \frac{^{12}C_1\, ^6C_5}{^{18}C_6} \cdot \frac{^{11}C_1\, ^1C_1}{^{12}C_2}}{\frac{^{12}C_2\, ^6C_4}{^{18}C_6} + \frac{^{12}C_1\, ^6C_5}{^{18}C_6} + \frac{^{12}C_0\, ^6C_6}{^{18}C_6}}$$

$$= \frac{^{12}C_2\, ^6C_4\, ^{10}C_1\, ^2C_1 + ^{12}C_1\, ^6C_5\, ^{11}C_1\, ^1C_1}{^{12}C_2 \left(^{12}C_2\, ^6C_4 + ^{12}C_1\, ^6C_5 + ^{12}C_0\, ^6C_6 \right)}$$

20. Sol.

A corresponds to one of A', B', C' and

B corresponds to one of the remaining of A', B', C' and

C corresponds to third of A', B', C'.

Hence six such permutations are possible

eg One of the permutations may $A \equiv A'$; $B \equiv B'$, $C \equiv C'$

From the given conditions:

A lies on L_1.

B lies on the line of intersection of P_1 and P_2

and 'C' lies on the line L_2 on the plane P_2.

Now, A' lies on $L_2 \equiv C$.

B' lies on the line of intersection of P_1 and $P_2 \equiv B$

C' lie on L_1 on plane $P_1 \equiv A$.

Hence there exist a particular set $[A', B', C']$ which is the permutation of $[A, B, C]$ such that both (i) and (ii) is satisfied. Here $[A', B', C'] \equiv [CBA]$.