JEE MAIN - 2004

MATHEMATICS

[Time : 2 hours]

General Instructions :

Note: Question number 1 to 10 carries **2 marks** each and 11 to 20 carries **4 marks** each.

1. Find the centre and radius of the circle formed by all the points represented by $z = x + iy$ satisfying the relation $\frac{|z-\alpha|}{|z-\beta|} = k\,(k \neq 1)$ where α and β are constant complex numbers given by $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$.

2. $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four distinct vectors satisfying the conditions $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then prove $\vec{a} \cdot \vec{b} + \vec{c} \cdot \vec{d} \neq \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{d}$.

3. Using permutation or otherwise prove that $\frac{n^2!}{(n!)^n}$ is an integer, where n is a positive integer.

4. If M is a 3×3 matrix, where $M^T M = I$ and det $(M) = 1$ then prove that det $(M - I) = 0$.

5. If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x . \cos\sqrt{\theta}}{1 + \sin^2\sqrt{\theta}} d\theta$ then find $\frac{dy}{dx}$ at $x = \pi$.

6. T is a parallelopiped in which A, B, C and D are vertices of one face. And the face just above it has corresponding vertices A', B', C', D'. T is now compressed to S with face $ABCD$ remaining same and A', B', C', D' shifted to A'', B'', C'', D'' in S. The volume of parallelopiped S is reduced to 90% of T. Prove that locus of A'' is a plane.

7. If $f:[-1,1]\to R$ and $f'(0)=\lim_{n\to\infty} nf\left(\frac{1}{n}\right)$ and $f(0)=0$. Find the value of $\lim_{n\to\infty}\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n$. Given that $0<\left|\lim_{n\to\infty}\cos^{-1}\left(\frac{1}{n}\right)\right|<\frac{\pi}{2}$.

8. If $p(x)=51x^{101}-2323x^{100}-45x+1035$, using Rolle's Theorem, prove that atleast one root lies between $\left(45^{1/100}, 46.\right)$

9. A plane is parallel to two lines whose direction ratios are $(1,0,-1)$ and $(-1,1,0)$ and it contains the point $(1,1,1)$. If it cuts coordinate axis at A, B, C, then find the volume of the tetrahedron $OABC$.

10. If A and B are two independent events, prove that $P(A\cup B).P(A'\cup B')\le P(C)$ where C is an event defined that exactly one of A and B occurs.

11. A curve passes through $(2,0)$ and the slope of tangent at point $P(x, y)$ equals $\frac{(x+1)^2+y-3}{(x+1)}$. Find the equation of the curve and area enclosed by the curve and the x-axis in the fourth quadrant.

12. A circle touches the line $2x+3y+1=0$ at the point $(1,-1)$ and is orthogonal to the circle which has the line segment having end points $(0,-1)$ and $(-2,3)$ as the diameter.

13. At any point P on the parabola $y^2-2y-4x+5=0$ a tangent is drawn which meets the directrix at Q. Find the locus of point R which divides QP externally in the ratio $\frac{1}{2}:1$

14. Evaluate $\int_{-\pi/3}^{\pi/3} \frac{\pi+4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx.$

15. If a, b, c are positive real numbers, then prove that $\left[(1+a)(1+b)(1+c)\right]^7 > 7^7 a^4 b^4 c^4$.

16. $f(x)=\begin{cases} b\sin^{-1}\left(\frac{x+c}{2}\right), & -\frac{1}{2}<x<0 \\ \frac{1}{2}, & x=0 \\ \frac{e^{\frac{a}{2}x}-1}{x}, & 0<x<\frac{1}{2} \end{cases}$

If $f(x)$ is differentiable at $x=0$ and $|c|<\frac{1}{2}$ then find the value of 'a' and prove that $64b^2=\left(4-c^2\right)$.

17. Prove that sin $x+2x \geq \frac{3x.(x+1)}{\pi}\ \forall x\in\left[0,\frac{\pi}{2}\right]$. (Justify the inequality, if any used).

18. $A=\begin{bmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{bmatrix}, B=\begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}, \mathrm{U}=\begin{bmatrix} f \\ g \\ h \end{bmatrix}, V=\begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$. If there is vector matrix X, such that $AX=U$ has infinitely many solutions, then prove that $BX=V$ cannot have a unique solution. If $afd\neq 0$ then prove that $BX=V$ has no solution.

19. A bag contains 12 red balls and 6 white balls. Six balls are drawn one by one without replacement of which atleast 4 balls are white. Find the probability that in the next two draws exactly one white ball is drawn.

(leave the answer in terms of ${}^nC_r$).

20. Two planes P_1 and P_2 pass through origin. Two lines L_1 and L_2 also passing through origin are such that L_1 lies on P_1 but not on P_2, L_2 lies on P_2 but not on P_1. A, B, C are three points other than origin, then prove that the permutation $[A', B', C']$ of $[A, B, C]$ exists such that

(i). A lies on L_1, B lies on P_1 not on L_1, C does not lie on P_1.

(ii). A' lies on L_2, B' lies on P_2 not on L_2, C' does not lie on P_2.