JEE MAIN-2005

MATHEMATICS

Q1. **Sol.**

Let C, S, B, T be the events of the person going by car, scooter, bus or train respectively.

Given that $P(C)=\frac{1}{7}, P(S)=\frac{3}{7}, P(B)=\frac{2}{7}, P(T)=\frac{1}{7}$

Let $\bar{L}$ be the event of the person reaching the office in time.

$$\Rightarrow P\left(\frac{\bar{L}}{C}\right)=\frac{7}{9}, P\left(\frac{\bar{L}}{S}\right)=\frac{8}{9}, P\left(\frac{\bar{L}}{B}\right)=\frac{5}{9}, P\left(\frac{\bar{L}}{T}\right)=\frac{8}{9}$$

$$\Rightarrow P\left(\frac{C}{\bar{L}}\right)=\frac{P\left(\frac{\bar{L}}{C}\right).P(C)}{P(\bar{L})}=\frac{\frac{1}{7}\times\frac{7}{9}}{\frac{1}{7}\times\frac{7}{9}+\frac{3}{7}\times\frac{8}{9}+\frac{2}{7}\times\frac{5}{9}+\frac{8}{9}\times\frac{1}{7}}=\frac{1}{7}.$$

Q2. **Sol.**

Let $y=2\sin t$

So,

$$y=\frac{1-2x+5x^2}{3x^2-2x-1}$$

$$\Rightarrow (3y-5)x^2-2x(y-1)-(y+1)=0$$

since $x \in -\left\{1,-\frac{1}{3}\right\}$, so $D \geq 0$

$$\Rightarrow y^2-y-1 \geq 0$$

or $y \geq \frac{1+\sqrt{5}}{2}$ and $y \leq \frac{1-\sqrt{5}}{2}$

or $\sin t \geq \frac{1+\sqrt{5}}{4}$ and $\sin t \leq \frac{1-\sqrt{5}}{4}$

Hence range of t is $\left[-\frac{\pi}{2},-\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10},-\frac{\pi}{2}\right]$.

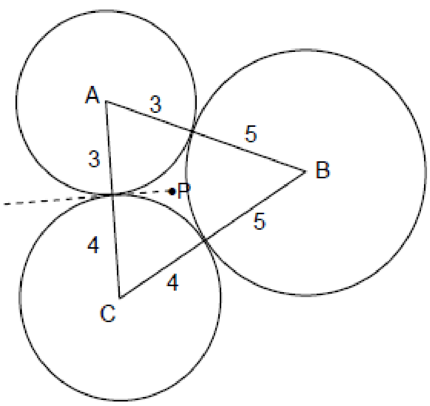
Q3. Sol.

Let A, B, C be the centre of the three circles. Clearly the point P is the in – centre of the ΔABC, and Hence

$$r=\frac{\Delta}{s}=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

Now $2s=7+8+9=24 \Rightarrow s=12$.

Hence $r=\sqrt{\frac{5.4.3}{12}}=\sqrt{5}$.



Q4. Sol.

Let the equation of plane be $(3\lambda+2)x+(\lambda-1)y+(\lambda+1)z-5\lambda-3=0$

$$\Rightarrow\left|\frac{6\lambda+4+\lambda-1-\lambda-1-5\lambda-3}{\sqrt{(3\lambda+2)^2+(\lambda-1)^2+(\lambda+1)^2}}\right|=\frac{1}{\sqrt{6}}$$

$$\Rightarrow 6(\lambda-1)^2=11\lambda^2+12\lambda+6\Rightarrow\lambda=0,-\frac{24}{5}.$$

$\Rightarrow$ The planes are $2x-y+z-3=$ and $62x+29y+19z-105=0.$

Q5. Sol.

$$|f(x_1)-f(x_2)|<(x_1-x_2)^2$$

$$\Rightarrow\lim_{x_1\to x_2}\left|\frac{f(x_1)-f(x_2)}{x_1-x_2}\right|<\lim_{x_1\to x_2}|x_1-x_2|\Rightarrow|f'(x)|<\delta\Rightarrow f'(x)=0$$

Hence $f(x)$ is a constant function and $P(1,2)$ lies on the curve.

$\Rightarrow f(x)=2$ is the curve.

Hence the equation of tangent is $y-2=0$.

Q6. Sol.

Let $S_n=\sum_{k=1}^{n}k.2^{n+1-k}=2^{n+1}=2^{n+1}\sum_{k=1}^{n}k.2^{-k}=2^{n+1}.2\left[1-\frac{1}{2^n}-\frac{n}{2^{n+1}}\right]$ (sum of the $A.G.P.$)

$$=2\left[2^{n+1}-2-n\right]$$

$$\Rightarrow\frac{n+1}{4}=2\Rightarrow n=7.$$

Q7. Sol.

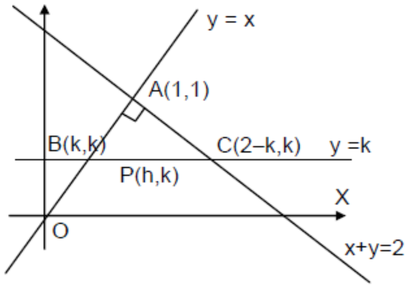
Area of triangle $= \frac{1}{2}.\ AB.AC = 4h^2$

and $AB = \sqrt{2}\,|k-1| = AC$

$\Rightarrow 4h^2 = \frac{1}{2}.2.(k-1)^2$

$\Rightarrow k-1 = \pm 2h.$

$\Rightarrow$ locus is $y = 2x+1, y = -2x+1.$



Q.8. Sol.

$$I = \int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\, dx$$

$$= \int_0^{2/\pi} e^{\cos x}\sin x\cos\left(\frac{1}{2}\cos x\right)dx \quad \left(\because \int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x)\end{cases}\right)$$

Let $\cos x = t$

$$I = 6\int_0^1 e^t \cos\left(\frac{t}{2}\right)$$

$$= \frac{24}{5}\left(e\cos\left(\frac{1}{2}\right) + \frac{e}{2}\sin\left(\frac{1}{2}\right) - 1\right).$$

Q9. Sol.

$\hat{v}$ is unit vector along the incident ray and $\hat{w}$ is the unit vector along the reflected ray. Hence $\hat{a}$ is a unit vector along the external bisector of $\hat{v}$ and $\hat{w}$. Hence

$$\hat{w}-\hat{v}=\lambda\hat{a}$$

$$\Rightarrow 1+1-\hat{w}\cdot\hat{v}=\lambda^2$$

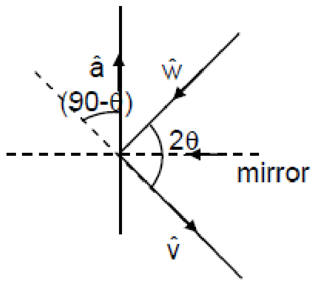
$$\text{or } 2-2\cos 2\theta=\lambda^2$$

$$\text{or } \lambda=2\sin\theta$$

where 2θ is the angle between $\hat{v}$ and $\hat{w}$.

Hence $\hat{w}-\hat{v}=2\sin\theta\hat{a}=2\cos\left(90^\circ-\theta\right)\hat{a}=-\left(2\hat{a}\cdot\hat{v}\right)\hat{a}$

$$\Rightarrow \hat{w}=\hat{v}-2\left(\hat{a}\cdot\hat{v}\right)\hat{a}.$$



Q10. Sol.

Any point on the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ is $\left(3\sec\theta, 2\tan\theta\right)$.

Chord of contact of the circle $x^2+y^2=9$ with respect to the point $\left(3\sec\theta, 2\tan\theta\right)$ is $3\sec\theta.x+2\tan\theta.y=9 \quad \ldots(1)$

Let $\left(x_1, y_1\right)$ be the mid – point of the chord of contact.

$\Rightarrow$ equation of chord in mid – point form is $xx_1 + yy_1 = x_1^2 + y_1^2 \quad \ldots(2)$

Since (1) and (2) represent the same line,

$$\frac{3\sec\theta}{x_1} = \frac{2\tan\theta}{y_1} = \frac{9}{z_1^2 + y_1^2}$$

$$\Rightarrow \sec\theta = \frac{9x_1}{3\left(x_1^2 + y_1^2\right)}, \tan\theta = \frac{9y_1}{2\left(x_1^2 + y_1^2\right)}$$

$$\text{Hence} \frac{81x_1^2}{9\left(x_1^2 + y_1^2\right)}, -\frac{81y_1^2}{4\left(x_1^2 + y_1^2\right)} = 1$$

$$\Rightarrow \text{the required locus is} \frac{x^2}{9} - \frac{y^2}{4} = \left(\frac{x^2 + y^2}{9}\right)^2.$$

Q11. **Sol**. Let the equations of tangents to the given circle and the ellipse respectively be $y = mx + 4\sqrt{1+m^2}$ *and* $y = mx + \sqrt{25m^2+4}$ Since both of these represent the same common tangent,

$$4\sqrt{1+m^2} = \sqrt{25m^2+4}$$

$$\Rightarrow 16\left(1+m^2\right) = 25m^2 + 4$$

$$\Rightarrow m = \pm\frac{2}{\sqrt{3}}$$

The tangent is at a point in the first quadrant $\Rightarrow m < 0$.

$\Rightarrow m = -\frac{2}{\sqrt{3}}$, so that the equation of the common tangent is

$$y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}.$$

It meets the coordinate axes at $A\left(2\sqrt{7}, 0\right)$ and $B\left(0, 4\sqrt{\frac{7}{3}}\right)$

$$\Rightarrow AB = \frac{14}{3}.$$

Q12. Sol.

Length of tangent $= \left| f\sqrt{1+\left(\frac{dx}{dy}\right)^2} \right| \Rightarrow 1 = y^2\left[1+\left(\frac{dx}{dy}\right)^2\right]$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{y}{\sqrt{1-y^2}} \Rightarrow \int \frac{\sqrt{1-y^2}}{y}dy = \pm x + c\,.$$

Writing $y=\sin\theta, dy=\cos\theta\, d\theta$ and integrating, we get the equation of the curve as

$$\sqrt{1-y^2} + \ln\left|\frac{1-\sqrt{1-y^2}}{y}\right| = \pm x + c\,.$$

Q13. Sol.

The region bounded by the given curves

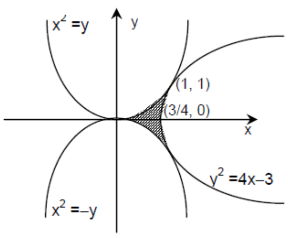
$x^2 = y, x^2 = -y$ and $y^2 = 4x-3$ is symmetrical about the x–axis. The parabolas $x^2 = y$ and $y^2 = 4x-3$ touch at the point $(1,1)$. Moreover the vertex of the curve

$y^2 = 4x-3$ is at $\left(\frac{3}{4},0\right)$.

Hence the area of the region

$$= 2\left[\int_0^1 x^2dx - \int_{3/4}^{1}\sqrt{4x-3}dx\right]$$

$$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \frac{1}{6}\left((4x-3)^{3/2}\right)_{3/4}^{1}\right] = 2\left[\frac{1}{3}-\frac{1}{6}\right] = \frac{1}{3}\text{sq. units.}$$



Q14. Sol.

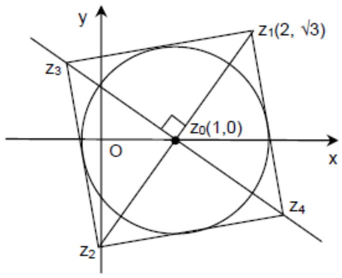
Since centre of circle i.e. $(1,0)$ is also the mid – point of diagonals of square

$$\Rightarrow \frac{z_1+z_2}{2}=z_0 \Rightarrow z_2=-\sqrt{3}i$$

and $\frac{z_3-1}{z_1-1}=e^{\pm i\pi/2}$

$\Rightarrow$ other vertices are

$$z_3, z_4=\left(1-\sqrt{3}\right)+i \text{ and } \left(1+\sqrt{3}\right)-i$$



Q15. Sol.

$$f(x-y)=f(x)g(y)-f(y)g(x) \quad \ldots(1)$$

Put $x=y$ in (1), we get

$f(0)=0$

Put $y=0$ in (1), we get

$g(0)=1.$

Now, $f'(0^+)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{f(0)g(-h)-g(0)f(-h)-f(0)}{h}$

$$=\lim_{n\to 0^+}\frac{f(-h)}{-h} \quad (\because f(0)=0)$$

$$=\lim_{n\to 0^+}\frac{f(0-h)-f(0)}{-h}$$

$$=f'(0^-).$$

Hence $f(x)$ is differentiable at $x=0$.

Put $y=x$ in $g(x-y)=g(x).\,g(y)+f(x).\,f(y)$.

Also $f^2(x)+g^2(x)=1$

$\Rightarrow g^2(x)=1-f^2(x)$

$\Rightarrow 2g'(0)g(0)=-2f(0)\,f'(0)=0 \Rightarrow g'(0)=0$.

Q16. Sol.

Let the polynomial be $P(x)=ax^3+bx^2+cx+d$

According to given conditions

$P(-1)=-a+b-c+d=10$

$P(1)=a+b-c+d=-6$

Also $P'(-1)=3a-2b+c=0$

and $P''(1)=6a+2b=0 \Rightarrow 3a+b=0$

Solving for $a,\ b,\ c,\ d$ we get

$P(x)=x^3-3x^2-9x+5$

$\Rightarrow P'(x)=3x^2-6x-9=3(x+1)(x-3)$

$\Rightarrow x=-1$is the point of maximum and $x=-3$is the point of minimum.

Hence distance between $(-,10)$ and $(3,-22)$ is $4\sqrt{65}$ units.

Q17. Sol.

Let us suppose that both $g(x)$ and $g''(x)$ are positive for all $x \in (-3,3)$.

Since $f^2(0)+g^2=9$ and $-1 \le f(x) \le 1, 2\sqrt{2} \le g(0) \le 3.$

From $f'(x)=g(x)$, we get

$$f(x)=\int_{-3}^{x} g(x)dx+f(-3).$$

Moreover, $g''(x)$ is assumed to be positive

$\Rightarrow$ the curve $y=g(x)$ is open upwards.

If $g(x)$ is decreasing, then for some value of

$x\int_{-3}^{x} g(x)dx >$ area of the rectangle $(0-(-3))2\sqrt{2}$

$\Rightarrow f(x) > 2\sqrt{2}\times 3-1$ i.e. $f(x)>1$ which is a contradiction.

If $g(x)$ is increasing, for some value of

$x\int_{-3}^{x} g(x)dx >$ area of the rectangle $(3-(-3))2\sqrt{2}$

$\Rightarrow f(x) > 2\sqrt{2}\times 3-1$ i.e. $f(x)>1$ which is a contradiction.

If $g(x)$ is minimum at $x=0$, then $\int_{-3}^{x} g(x)dx >$ area of the rectangle $(3-0)2\sqrt{2}$

$\Rightarrow f(x) > 2\sqrt{2}\times 6-1$ i.e. $f(x)>1$ which is a contradiction.

Hence $g(x)$ and $g''(x)$ cannot be both positive throughout the interval $-3,3$.

Similarly we can prove that $g(x)$ and $g''(x)$ cannot be both negative throughout the interval $-3,3$.

Hence there is atleast one value of $c \in (-3,3)$ where $g(x)$ and $g''(x)$ are of opposite sign $\Rightarrow g(c).g''(c) < 0.$

Alternate:

$$\int_0^3 g(x)\,dx = \int_0^3 f'(x)\,dx = f(3) - f(0)$$

$$\Rightarrow \left|\int_0^3 g(x)\,dx\right| < 2 \quad \ldots\ldots(1)$$

In the same way $\left|\int_{-3}^0 g(x)\,dx\right| < 2 \quad \ldots\ldots(2)$

$$\Rightarrow \left|\int_0^3 g(x)\,dx\right| + \left|\int_{-3}^0 g(x)\,dx\right| < 4 \quad \ldots\ldots(3)$$

From $(f(0))^2 + (g(0))^2 = 9$

we get

$2\sqrt{2} < g(0) < 3 \quad \ldots\ldots(4)$

or $-3 < g(0 < -2\sqrt{2}) \quad \ldots\ldots(5)$

Case I: $2\sqrt{2} < g(0) < 3$

Let $g(x)$ is concave upward $\forall x(-3,3)$ then the area

$$\left|\int_{-3}^0 g(x)\,dx\right| + \left|\int_0^3 g(x)\,dx\right| < 6\sqrt{2}$$

which is a contradiction from equation (3).

$\therefore g(x)$ will be concave downward for some $c \in (-3,3)$ i.e. $g''(c) < 0 \quad \ldots(6)$

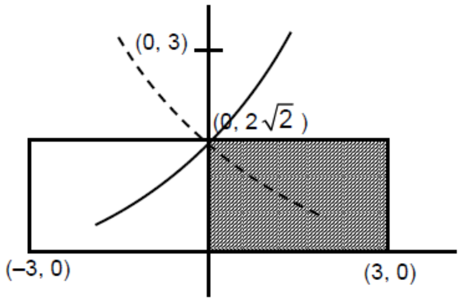
also at that point c

$g(c)$ will be greater than $2\sqrt{2}$

$\Rightarrow g(c) > 0 \quad \ldots(7)$

From equation (6) and (7)

$g(c).\ g''(c) < 0$ for some $c \in (-3,3)$.



Case II: $-3 < g(0) < -2\sqrt{2}$

Let $g(x)$ is concave downward $\forall x(-3,3)$ then the area

$$\left|\int_{-3}^{0} g(x)\,dx\right| + \left|\int_{0}^{3} g(x)\,dx\right| < 6\sqrt{2}$$

which is a contradiction from equation (3).

$\therefore g(x)$ will be concave upward for some $c \in (-3,3)$ i.e. $g''(c) < 0 \quad \ldots(8)$

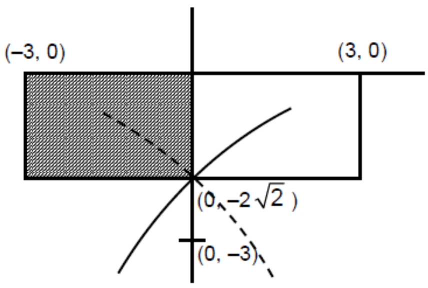
also at that point c

$g(c)$ will be less than $2\sqrt{2}$

$\Rightarrow g(c) < 0 \quad \ldots(9)$

From equation (8) and (9)

$g(c).\ g''(c) < 0$ for some $c \in (-3,3)$.



Q18. Sol.

Let we have

$$4a^2 f(-1)+4a\,f(1)+f(2)=3a^2+3a \quad \ldots(1)$$

$$4b^2 f(-1)+4b\,f(1)+f(2)=3b^2+3b \quad \ldots(2)$$

$$4c^2 f(-1)+4c\,f(1)+f(2)=3c^2+3c \quad \ldots(3)$$

Consider a quadratic equation

$$4x^2 f(-1)+4xf(1)+f(2)=3x^2+3x$$

$$\text{or}\left[4f(-1)-3\right]x^2+\left[4f(1)-3\right]x+f(2)=0 \quad \ldots(4)$$

As equation (4) has three roots i.e. $x=a,\ b,\ c$. It is an identity.

$$\Rightarrow f(-1)=\frac{3}{4},\ f(1)=\frac{3}{4}\text{and } f(2)=0$$

$$\Rightarrow f(x)=\frac{\left(4-x^2\right)}{4} \quad \ldots(5)$$

Let point A be $(-2,0)$ and B be $\left(2t,-t^2+1\right)$

Now as AB subtends a right angle at the vertex

$V(0,1)$

$$\frac{1}{2}\times\frac{-t^2}{2t}=-1\Rightarrow t=4$$

$$\Rightarrow B\equiv(8,-15)$$

$$\therefore \text{Area}=\int_{-2}^{8}\left[\frac{4-x^2}{4}+\frac{3x+6}{2}\right]dx=\frac{125}{3}\text{sq.units.}$$