IITJEE–2006

Mathematics

[Time allowed: 2 hours]

Note: Question number 1 to 12 carries (3, -1) ***marks*** each, 13 to 20 carries (5, -1) ***marks*** each, 21 to 32 carries (5, -2) ***marks*** each and 33 to 40 carries (6, 0) ***marks*** each.

Section – A (Single Option Correct)

1. For $x > 0, \lim_{x \to 0}\left((\sin x)^{1/x} + (1/x)^{\sin x}\right)$ is

(A) 0

(B) -1

(C) 1

(D) 2

2. $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}}dx$ is equal to

(A) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + c$

(B) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + c$

(C) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + c$

(D) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + c$

3. Given an isosceles triangle, whose one angle is $120°$ and radius of its incircle $=\sqrt{3}$. Then the area of the triangle in sq. units is

(A) $7+12\sqrt{3}$

(B) $12-7\sqrt{3}$

(C) $12+7\sqrt{3}$

(D) 4π

4. If $0<\theta<2\pi$, then the intervals of values of θ for which $2\sin^2\theta-5\sin\theta2>0$, is

(A) $\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},2\pi\right)$

(B) $\left(\frac{\pi}{8},\frac{5\pi}{6}\right)$

(C) $\left(0,\frac{\pi}{8}\right)\cup\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$

(D) $\left(\frac{41\pi}{48},\pi\right)$

5. If $w=\alpha+i\beta$, where $\beta\neq 0$ and $z\neq 1$, satisfies the condition that $\left(\frac{w-\bar{w}z}{1-z}\right)$ is purely real, then the set of values of z is

(A) $\{z:|z|=1\}$

(B) $\{z:z=\bar{z}\}$

(C) $\{z : z \neq 1\}$

(D) $\{z : |z| = 1, z \neq 1\}$

6. Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a+b+c)x + 3\lambda(ab+bc+ca) = 0$ are real, then

(A) $\lambda < \frac{4}{3}$

(B) $\lambda < \frac{5}{3}$

(C) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$

(D) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

7. If $f'(x) = -f(x)$ and $g(x) = f'(x)$ and $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and given that $F(5) = 5$, then $F(10)$ is equal to

(A) 5

(B) 10

(C) 0

(D) 15

8. If r, s, t are prime numbers and p, q are the positive integers such that the LCM of p, q is $r^2t^4s^2$, then the number of ordered pair (p, q) is

(A) 252

(B) 254

(C) 225

(D) 224

9. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{\cot\theta}, t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\tan\theta}$, then

(A) $t_1 > t_2 > t_3 > t_4$

(B) $t_4 > t_3 > t_1 > t_2$

(C) $t_3 > t_1 > t_2 > t_4$

(D) $t_2 > t_3 > t_1 > t_4$

10. The axis of a parabola is along the line $y = x$ and the distance of its vertex from origin is $\sqrt{2}$ and that from its focus is $2\sqrt{2}$. If vertex and focus both lie in the first quadrant, then the equation of the parabola is

(A) $(x+y)^2 = (x-y-2)$

(B) $(x-y)^2 = (x+y-2)$

(C) $(x-y)^2 = 4(x+y-2)$

(D) $(x-y)^2 = 8(x+y-2)$

11. A plane passes through $(1,-2,1)$ and is perpendicular to two planes $2x-2y+z=0$ and $x-y+2z=4$. The distance of the plane from the point $(1,2,2)$ is

(A) 0

(B) 1

(C) $\sqrt{2}$

(D) $2\sqrt{2}$

12. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is

(A) $4\hat{i}-\hat{j}+4\hat{k}$

(B) $3\hat{i}+\hat{j}-3\hat{k}$

(C) $2\hat{i}+\hat{j}-2\hat{k}$

(D) $4\hat{i}+\hat{j}-4\hat{k}$

Section – B (May have more than one option correct)

13. The equations of the common tangents to the parabola $y=x^2$ and $y=-(x-2)^2$ is/are

(A) $y=4(x-1)$

(B) $y=0$

(C) $y=-4(x-1)$

(D) $y=-30x-50$

14. If $f(x)=\min\{1,x^2,x^3\}$, then

(A) $f(x)$ is continuous $\forall x\in R$

(B) $f(x)>0, \forall x>1$

(C) $f(x)$ is not differentiable but continuous $\forall x\in R$

(D) $f(x)$ is not differentiable for two values of x

15. A tangent drawn to the curve $y=f(x)$ at $P(x,y)$ cuts the x-axis and y-axis at A and B respectively such that $BP:AP=3:1$, given that $f(1)=1$, then

(A) equation of curve is $x\frac{dy}{dx}-3y=0$

(B) normal at $(1,1)$ is $x+3y=4$

(C) curve passes through $(2,1/8)$

(D) equation of curve is $x\frac{dy}{dx}+3y=0$

16. If a hyperbola passes through the focus of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and its transverse and conjugate axes coincide with the major and minor axes of the ellipse, and the product of eccentricities is 1, then

(A) the equation of hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$

(B) the equation of hyperbola is $\frac{x^2}{9}-\frac{y^2}{25}=1$

(C) focus of hyperbola is $(5,0)$

(D) focus of hyperbola is $(5\sqrt{3},0)$

17. Internal bisector of $\angle A$ of triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a,b,c represent sides of ΔABC then

(A) AE is HM of b and c

(B) $AD=\frac{2bc}{b+c}\cos\frac{A}{2}$

(C) $EF=\frac{4bc}{b+c}\sin\frac{A}{2}$

(D) the triangle AEF is isosceles

18. $f(x)$ is cubic polynomial which has local maximum at $x=-1$. If $f(2)=18$, $f(1)=-1$ and $f'(x)$ has local minima at $x=0$, then

(A) the distance between $(-1,2)$ and $(a,\text{f}(a))$, where $x=a$ is the point of local minima is $2\sqrt{5}$

(B) $f(x)$ is increasing for $x\in\left[1,2\sqrt{5}\right]$

(C) $f(x)$ has local minima at $x=1$

(D) the value of $f(0)=5$

19. Let $\bar{A}$ be vector parallel to line of intersection of planes P_1 and P_2 through origin. P_1 is parallel to the vectors $2\hat{j}+3\hat{k}$ and $4\hat{j}-3\hat{k}$ and P_2 is parallel to $\hat{j}-\hat{k}$ and $3\hat{i}+3\hat{j}$, then the angle between vectors $\bar{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$

(D) $\frac{3\pi}{4}$

20. $f(x)=\begin{cases} e^x & 0\le x\le 1 \\ 2-e^{x-1} & 1<x\le 2 \\ x-e & 2<x\le 3 \end{cases}$ and $g(x)=\int_0^x f(t)\,dt, x\in[1,3]$ then $g(x)$ has

(A) local maxima at $x=1+\ln 2$ and local minima at $x=e$

(B) local maxima at $x=1$ and local minima at $x=2$

(C) no local maxima

(D) no local minima

Section − C

Comprehension I

There are n urns each containing $n+1$ balls such that the ith urn contains i white balls and $(n+1-i)$ red balls. Let u_i be the event of selecting ith urn, $i=1,2,2\ldots,n$ and w denotes the event of getting a white ball.

21. If $p(u_i)\propto i$, where $i=1,2,3,\ldots n,$ then $\lim\limits_{n\to\infty}P(w)$ is equal to

(A) 1

(B) $\frac{2}{3}$

(C) $\frac{3}{4}$

(D) $\frac{1}{4}$

22. If $P(u_i)=c$ where c is a constant then $P(u_n/w)$ is equal to

(A) $\frac{2}{n+1}$

(B) $\frac{1}{n+1}$

(C) $\frac{n}{n+1}$

(D) $\frac{1}{2}$

23. If n is even and E denotes the event of choosing even numbered urn $\left(P(u_i)=\frac{1}{n}\right)$, then the value of $P(w/E)$ is

(A) $\frac{n+2}{2n+1}$

(B) $\frac{n+2}{2(n+1)}$

(C) $\frac{n}{n+1}$

(D) $\frac{1}{n+1}$

24. $\int_0^{\pi/2} \sin x\, dx$ is equal to

(A) $\frac{\pi}{8}\left(1+\sqrt{2}\right)$

(B) $\frac{\pi}{4}\left(1+\sqrt{2}\right)$

(C) $\frac{\pi}{8\sqrt{2}}$

(D) $\frac{\pi}{4\sqrt{2}}$

25. If $f'(x) < 0 \forall x \in (a,b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to

(A) $\frac{f(b)-f(a)}{b-a}$

(B) $\frac{2(f(b)-f(a))}{b-a}$

(C) $\frac{2f(b)-f(a)}{2b-a}$

(D) 0

Comprehension III

Let ABCD be a square of side length 2 units. C2 is the circle through vertices A, B, C, D and C1 is the circle touching all the sides ***of the square ABCD. L is a line through A.***

26. If P is a point on C_1 and Q in another point on C_2, then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to

(A) 0.75

(B) 1.25

(C) 1

(D) 0.5

27. A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

(A) ellipse

(B) hyperbola

(C) parabola

(D) parts of straight line

28. A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is

(A) $\frac{1}{2}$ sq. units

(B) $\frac{2}{3}$ sq. units

(C) 1 sq. units

(D) 2 sq. units

Comprehension IV

$A=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}$, if U_1, U_2 and U_3 are columns matrices satisfying.

$AU_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, AU_2=\begin{bmatrix}2\\3\\0\end{bmatrix}, AU_3=\begin{bmatrix}2\\3\\1\end{bmatrix}$ and U is 3×3 matrix whose columns are U_1, U_2, U_3 then answer the following questions

29. The value of $|U|$ is

(A) 3

(B) −3

(C) 3/2

(D) 2

30. The sum of the elements of U^{-1} is

(A) −1

(B) 0

(C) 1

(D) 3

31. The value of $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is

(A) 5

(B) 5/2

(C) 4

(D) 3/2

Section – D

32. If roots of the equation $x^2 - 10cx - 11d = 0$ are a, b and those of $x^2 - 10ax - 11b = 0$ are c, d, then the value of $a+b+c+d$ is (a, b, c and d are distinct numbers)

33. The value of $5050 \dfrac{\int_0^1 \left(1-x^{50}\right)^{100} dx}{\int_0^1 \left(1-x^{50}\right)^{101} dx}$ is

34. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$ then find the minimum natural number n_0 such that $b_n > a_n \forall n > n_0$

35. If $f(x)$ is a twice differentiable function such that $f(a)=0, f(b)=2, f(c)=-, f(d)=2, f(e)=0$, where $a<b<c<d<e$, then the minimum number of zeroes of $g(x) = \left(f'(x)\right)^2 + f'(x) f(x)$ in the interval $[a, e]$ is

Section – E

36. Match the following:

Normals are drawn at points P, Q and R lying on the parabola y2 = 4x which intersect at (3, 0). Then

(i)	Area of ΔPQR	(A)	2
(ii)	Radius of circumcircle of ΔPQR	(B)	$5/2$
(iii)	Centroid of ΔPQR	(C)	$(5/2,0)$
(iv)	Circumcentre of ΔPQR	(D)	$(2/3,0)$

37. Match the following

(i)	$\int_0^{\pi/2}(\sin x)^{\cos x}\left(\cos x\cot x-\log(\sin x)^{\sin x}\right)dx$	(A)	1
(ii)	Area bounded by $-4y^2=x$ and $x-1=-5y^2$	(B)	0
(iii)	Cosine of the angle of intersection of curves $y=3^{x-1}\log x$ and $y=x^x-1$ is	(C)	$6\ln 2$
(iv)	Data could not be retrieved.	(D)	$4/3$

38. Match the following

(i)	Two rays in the first quadrant $x+y=\|a\|$ and $ax-y=1$ intersects each other in the interval $a\in(a_0,\infty)$, the value of a_0 is	(A)	2
(ii)	Point (α,β,γ) lies on the plane $x+y+z=2$. Let $\vec{a}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}, \hat{k}\times(\hat{k}\times\vec{a})=0$, then $\gamma=$.	(B)	4/3
(iii)	$\left\|\int_0^1(1-y^2)dy\right\|+\left\|\int_1^0(y^2-1)dy\right\|$	(C)	$\left\|\int_0^1\sqrt{1-x}\,dx\right\|+\left\|\int_{-1}^0\sqrt{1+x}\,dx\right\|$
(iv)	If $\sin A\sin B\sin C+\cos A\cos B=1$, then the value of $\sin C=$	(D)	1

39. Match the following

(i)	$\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{1}{2i^2}\right)=t$, then $\tan t=$	(A)	0
(ii)	Sides a,b,c of a triangle ABC are in AP and $\cos\theta_1=\frac{a}{b+c},\cos\theta_2=\frac{b}{a+c},\cos\theta_3=\frac{c}{a+b}$, then $\tan^2\left(\frac{\theta_1}{2}\right)+\tan^2\left(\frac{\theta_3}{2}\right)=$	(B)	1
(iii)	A line is perpendicular to $x+2y+2z=0$ and passes through $(0,1,0)$. The perpendicular distance of this line from the origin is	(C)	$\frac{\sqrt{5}}{3}$
(iv)	Data could not be retrieved.	(D)	2/3