IIT-JEE-2007

PAPER-II

MATHEMATICS

SECTION – I

45. Solution : (B)

Since $\vec{a}, \vec{b}, \vec{c}$ are unit vectors and $\vec{a} + \vec{b} + \vec{c} = 0$,

$\vec{a}, \vec{b}, \vec{c}$ represent an equilateral triangle.

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \neq \vec{0}$

46. Solution : (A)

$$Here\ \ ff(x) = \frac{f(x)}{\left[1 + f(x)^n\right]^{1/n}} = \frac{x}{\left(1 + 2x^n\right)^{1/n}}$$

$$ff(x) = \frac{f(x)}{\left(1 + 3x^n\right)^{1/n}}$$

$$\Rightarrow g(x) = \Big(\underbrace{fofo \ldots of}_{n\,terms}\Big)(x) = \frac{x}{\left(1 + nx^n\right)^{1/n}}$$

$$Hence\ I \int x^{n-2} g(x)\,dx = \int \frac{x^{n-1}dx}{\left(1 + nx^n\right)^{1/n}}$$

$$= \frac{1}{n^2} \int \frac{n^2 x^{n-1} dx}{\left(1 + nx^n\right)^{1/n}} = \frac{1}{n^2} \int \frac{\frac{d}{dx}\left(1 + nx^n\right)}{\left(1 + nx^n\right)^{1/n}} dx$$

$$\therefore I = \frac{1}{n(n-1)} \left(1 + nx^n\right)^{1 - \frac{1}{n}} + K$$

47. Solution : (D)

Since, $\frac{dx}{dy}=\frac{1}{dy/dx}=\left(\frac{dy}{dx}\right)^{-1}$

$$\Rightarrow \frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1}\frac{dx}{dy}$$

$$\Rightarrow \frac{d^2x}{dy^2}=-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}\left(\frac{dx}{dy}\right)=-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

***48. Solution : (C)**

COCHIN

The second place can be filled in ${}^4C_1$ ways and the remaining four alphabets can be arranged in $4!$ ways in four different places. The next 97^{th} word will be *COCHIN*. Hence, there are 96 words before *COCHIN*

***49. Solution : (D)**

$Let\ z=\cos\theta+\sin\theta,\ so\,that$

$$\frac{z}{1-z^2}=\frac{\cos\theta+\sin\theta}{1-(\cos 2\theta+i\sin 2\theta)}$$

$$=\frac{\cos\theta+i\sin\theta}{2i\sin^2\theta-2i\sin\theta\cos\theta}=\frac{\cos\theta+i\sin\theta}{-2i\sin\theta(\cos\theta+i\sin\theta)}$$

$$=\frac{i}{2\sin\theta}$$

Hence $\frac{z}{1-z^2}$ lies on the imaginary axis i.e., $x=0$.

Alternative

Let $E=\frac{z}{1-z^2}=\frac{z}{z\bar{z}-z^2}=\frac{z}{\bar{z}-z}$

which is imaginary.

***50. Solution : (B)**

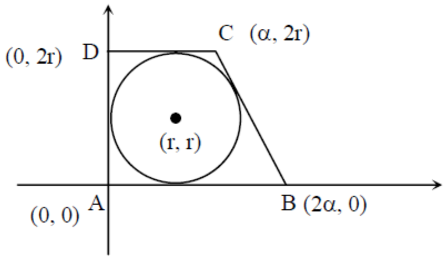
$18 = \frac{1}{2}(3\alpha)(2r) \Rightarrow \alpha r = 6$

$Line\ y = -\frac{2r}{\alpha}(x - 2\alpha)$ is tangent to $(x-r)^2 +$

$(y-r)^2 = r^2$

$2\alpha = 3r\ and\ \alpha r = 6$

$r = 2$



Alternate

$\frac{1}{2}(x + 2x) \times 2r = 18$

$xr = 6 \quad \text{K } (1)$

$\tan\theta = \frac{x-r}{r} \quad \tan(90-\theta) = \frac{2x-r}{r}$

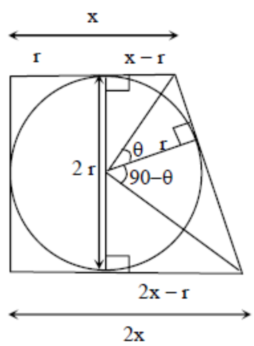
$\frac{x-r}{r} = \frac{r}{2x-r}$

$x(2x - 3r) = 0$

$x = \frac{3r}{2} \quad \text{K } (2)$

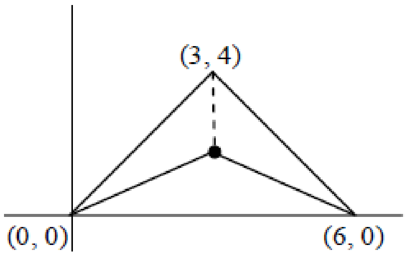
$From\,(1)\ and\,(2)$

$r = 2.$



***51. Solution : (C)**

Since, Δ is isosceles, hence centroid is the desired point.



52. Solution : (C)

$$\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{y}$$

$$\Rightarrow \int \frac{y}{\sqrt{1-y^2}}dy=\int dx$$

$$\Rightarrow -\sqrt{1-y^2}=x+c$$

$$\Rightarrow (x+c)^2+y^2=1$$

$$centre(-c,0); radius\sqrt{c^2-c^2+1}=1$$

53. Solution : (C)

$$P\left(\frac{E^c\cap F^c}{G}\right)=\frac{P(E^c\cap F^c\cap G)}{P(G)}=\frac{P(G)-P(E\cap G)-P(G\cap F)}{P(G)}$$

$$=\frac{P(G)(1-P(E)-P(F))}{P(G)} \quad [\therefore P(G)\neq 0]$$

$$=1=P(E)-P(F)$$

$$=P(E)-P(F)$$

SECTION −II

Assertion − Reason Type

54. Solution : (C)

$f(x) = 2 + \cos \forall x \in R$

Statement : 1

There exists a point $c \in [t, t+\pi]$ where $f'(c) = 0$

Hence, statement 1 is true.

Statement 2:

$f(t) = f(t+2\pi)$ is true.

But statement 2 is not a correct explanation for statement 1.

55. Solution : (D)

$3x - 6y - 2z = 15$

$2x + y - 2z = 5$

for $z = 0$, we get $x = 3, y = -1$

Direction vectors of plane are

$< 3 \quad -6 \quad -2 >$ and $< 2 \quad 1 \quad -2 >$

then the dr's of line of intersection of planes is $< 14 \quad 2 \quad 15 >$

$$\frac{x-3}{14} = \frac{y+1}{2} = \frac{z-0}{15} = \lambda$$

$\Rightarrow x = 14\lambda + 3 \quad y = 2\lambda - 1 \quad y = 15\lambda$

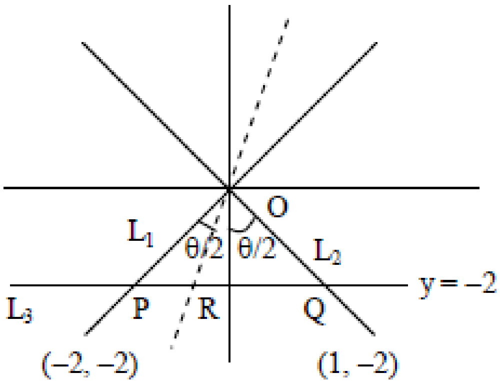
Hence, statement 1 is false.

But statement 2 is true.

***56. Solution : (D)**

In ΔOPQ

Clearly $\frac{PR}{RQ}=\frac{OP}{OQ}=\frac{2\sqrt{2}}{\sqrt{5}}$



***57. Solution : (D)**

$$y=-\frac{x^2}{2}+x+1$$

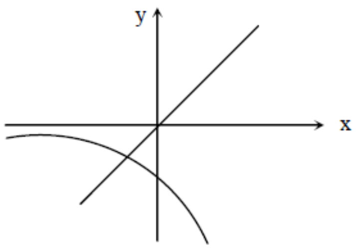
$$\Rightarrow y-\frac{3}{2}=-\frac{1}{2}-(x-1)^2$$

$$\Rightarrow it\ is\ symmetric\ about\ x=1$$

SECTION – III

58. Solution : (B)

Line $y = x$ intersect the curve $y = ke^x$ at exactly one point when $k \leq 0$.



59. Solution : (A)

$Let\ f(x) - ke^x - x$

$f'(x) - ke^x - l - x \Rightarrow x = -ln\,k$

$f''(x) - ke^x$

$f''(x)_{x=lnk} = 1 > 0$

For one root of given equation

$1 + \ln k = 0$

Hence $k = \frac{1}{e}$.

60. Solution : (A)

For two distinct roots $1 + \ln k < 0 (k > 0)$

$\ln k < -1$

$k < \frac{1}{e}$

hence $k \in \left(0, \frac{1}{e}\right)$

***61. Solution : (C)**

$$A_1 = \frac{a+b}{2}; G_1 = \sqrt{ab}; H_1 \frac{2ab}{a+b}$$

$$A_n, \frac{A_{n-1}+H_{n-1}}{2}, G_n = \sqrt{A_{n-1}H_{n-1}}, H_n = \frac{2A_{n-1}H_{n-1}}{A_{n-1}+H_{n-1}}$$

$$Clearly,\ G_1 = G_2 = G_3 = \mathrm{K} = \sqrt{ab}$$

***62. Solution : (B)**

$$A_2\ is\ A.M.\ of, A_1\ and\ A_1 > H_1 \Rightarrow A_1 > A_2 > H_1$$

$$A_3\ is\ A.M.\ of, A_2\ and\ A_2 > H_2 \Rightarrow A_2 > A_3 > H_2$$

$$\therefore A_2 > A_3 > A_3 > \mathrm{K}$$

***63. Solution : (B)**

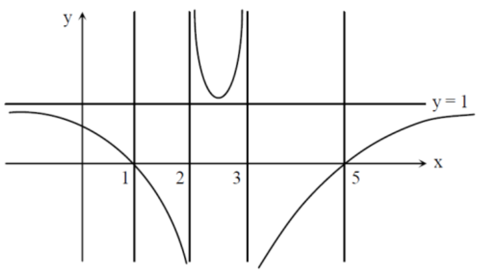
As above $A_1 > H_2 > H_1, A_2 > H_3 > H_2$

$$A_1 > H_2 > H_1, A_2 > H_3 > H_2$$

64. Solution : A → p, r, s ; B → q, s ; C → q, s ; D → p, r, s

$$f(x) = \frac{(x-1)(x-5)}{(x-2)(x-3)}$$

The graph of $f(x)$ is shown



(A) $\begin{aligned} &If\ -1 < x < 1 \\ &\Rightarrow 0 < f(x) < 1 \end{aligned}$

(B) $If\ 1 < x < 2 \Rightarrow f(x) < 0$

(C) $If\ 3 < x < 5 \Rightarrow f(x) < 0$

(D) $If\ x > 5 \Rightarrow 0 < f(x) < 1$

65. Solution : A → p ; B → q ; C → p ; D → s

(A) $\begin{aligned} &If\ a = 1,\ b = 0 \\ &then\ \sin^{-1} x + \cos^{-1} y = 0 \\ &\Rightarrow \sin^{-1} x = -\cos^{-1} y \end{aligned}$

(B) $\begin{aligned} &If\ a = 1\ and\ b = 1,\ then \\ &\sin^{-1} x + \cos^{-1} y + \cos^{-1} xy = \frac{\pi}{2} \\ &\Rightarrow \cos^{-1} x - \cos^{-1} y = \cos^{-1} xy \\ &\Rightarrow xy + \sqrt{1-x^2}\sqrt{1-y^2} = xy\ (taking\ \text{sine}\ on\ both\ sides) \end{aligned}$

(C) $\begin{aligned} &If\ a = 1,\ \ b = 2 \\ &\Rightarrow \sin^{-1} x + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2} \\ &\Rightarrow \sin^{-1} x + \cos^{-1} y = \sin^{-1}(2xy) \\ &\Rightarrow xy + \sqrt{1-x^2}\sqrt{1-y^2} = 2xy \\ &\Rightarrow x^2 + y^2\ (on\ squaring) \end{aligned}$

(D) $\begin{aligned} &If\ a = 2\ and\ b = 2,\ then \\ &\sin^{-1}(2x) + \cos^{-1}(y) + \cos^{-1}(2xy) = \frac{\pi}{2} \\ &\Rightarrow 2xy + \sqrt{1-4x^2}\sqrt{1-y^2} = 2xy \\ &\Rightarrow (4x^2 - 1)(y^2 - 1) = 0 \end{aligned}$

66. Solution : A → p, q ; B → p, q ; C → q, r ; D → q, r

(A) When two circles are intersecting they have a common normal and common tangent.

(B) Two mutually external circles have a common normal and common tangent.

(C) When one circle lies inside of other then, they have a common normal but no common tangent.

(D) Two branches of a hyperbola have a common normal but no common tangent.