JEE MAIN - 2008

MATHEMATICS

1. Sol. (B)

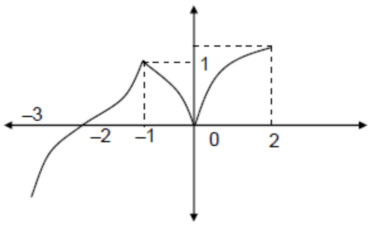
$$\left(ax^2+by^2+c\right)\left(x^2-5xy+6y^2\right)=0$$

$$\Rightarrow ax^2+by^2+c=0 \text{ or } x^2-5xy+6y^2=0$$

$$\Rightarrow x^2+y^2=\left(-\frac{c}{a}\right) \text{iff } a=b, x-2y=0 \text{ and } x-3y=0$$

Hence the given equation represents two straight lines and a circle, when $a=b$ and c is of sign opposite to that of a.

2. Sol. (C)

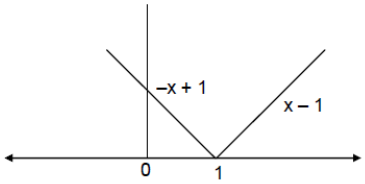


Local maximum at $x=-1$

and local minimum at $x=0$

Hence total number of local maxima and local minima is 2

3. Sol. (C)



From graph, $p=-1$

$$\Rightarrow \lim_{x\to 1^+} g(x) = -1$$

$$\Rightarrow \lim_{h\to 0} g(1+h) = -1$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{h^n}{\log\cos^m h}\right) = -1$$

$$\Rightarrow \lim_{h\to 0}\frac{n\cdot h^{n-1}}{m\cdot(-\tanh)} = -\left(\frac{n}{m}\right)\lim_{h\to 0}\left(\frac{h^{n-1}}{\tanh}\right) = -1,\ \text{which holds if } n = m = 2.$$

4. Sol. (C)

$$\sqrt{1+x^2}\left[\left(x\cos\cot^{-1}x + \sin\cot^{-1}x\right)^2 - 1\right]^{1/2}$$

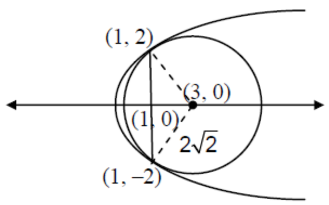
$$= \sqrt{1+x^2}\left[\left(x\cos\cos^{-1}\frac{x}{\sqrt{1+x^2}} + \sin\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}}\right)^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left(x^2+1-1\right)^{1/2} = x\sqrt{1+x^2}.$$

5. Sol. (B)

The circle and the parabola touch each other at $x = 1$ i.e. at the points $(1, 2)$ and $(1, -2)$ as shown in the figure.

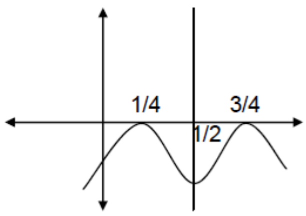


6. Sol. (A)

$$\text{Volume } = \left|\hat{a}\cdot\left(\hat{b}\times\hat{c}\right)\right| = \sqrt{\begin{vmatrix}\hat{a}\cdot\hat{a} & \hat{a}\cdot\hat{b} & \hat{a}\cdot\hat{c}\\ \hat{b}\cdot\hat{a} & \hat{b}\cdot\hat{b} & \hat{b}\cdot\hat{c}\\ \hat{c}\cdot\hat{a} & \hat{c}\cdot\hat{b} & \hat{c}\cdot\hat{c}\end{vmatrix}}$$

$$= \sqrt{\begin{vmatrix}1 & 1/2 & 1/2\\ 1/2 & 1 & 1/2\\ 1/2 & 1/2 & 1\end{vmatrix}} = \frac{1}{\sqrt{2}}.$$

7. Sol. (A, B, C, D)



$$f(x) = f(1-x)$$

Put $x = 1/2 + x$

$$f\left(\frac{1}{2}+x\right) = f\left(\frac{1}{2}-x\right)$$

Hence $f(x+1/2)$ is an even function or $f(x+1/2)\ \sin x$ is an odd function.

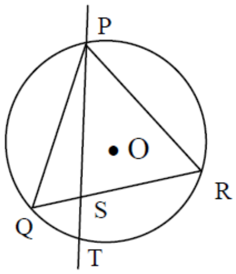
Also, $f(x) = -f'(1-x)$ and for $x = 1/2$, we have $f'(1/2) = 0$.

Also, $\int\limits_{1/2}^{1} f(1-t)e^{\sin \pi t}dt = -\int\limits_{1/2}^{0} f(y)e^{\sin \pi y}dy$ (obtained by putting, $1-t=y$).

Since $f'(1/4)=0,\ f'(3/4)=0.$ Also $f'(1/2)=0$

$\Rightarrow f'(x) = 0$ atleast twice in $[0,1]$ (Rolle's Theorem)

8. Sol. (B, D)



$$PS \times ST = QS \times SR$$

$$\frac{\frac{1}{PS}+\frac{1}{ST}}{2} > \sqrt{\frac{1}{PS}\times\frac{1}{ST}}$$

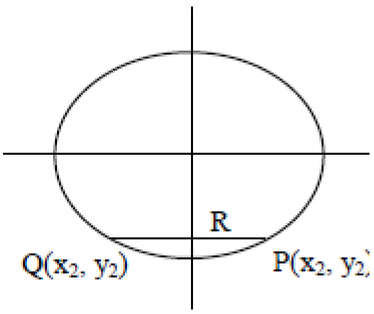
$$\Rightarrow \frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$$

$$\frac{QS+SR}{2} > \sqrt{QS \times SR}$$

$$\frac{QR}{2} > \sqrt{QS \times SR} \Rightarrow \frac{1}{\sqrt{QS \times SR}} > \frac{2}{QR}$$

$$\Rightarrow \frac{1}{PS}+\frac{1}{ST} > \frac{4}{QR}.$$

9. Sol. (B, C)



$$\frac{x^2}{4}+\frac{y^2}{1}=1$$

$b^2 = a^2(1-e^2)$

$\Rightarrow e = \dfrac{\sqrt{3}}{2}$

$\Rightarrow P\left(\sqrt{3},-\dfrac{1}{2}\right)$ and $\left(-\sqrt{3},-\dfrac{1}{2}\right)$ (given y_1 and y_2 less than 0)

Co-ordinates of mid-point of PQ are $R \equiv \left(0,-\dfrac{1}{2}\right)$.

$PQ = 2\sqrt{3} =$ length of latus rectum.

$\Rightarrow$ two parabola are possible whose vertices are $\left(0,-\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\right)$ and $\left(0,\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\right)$.

Hence the equations of the parabolas are $x^2 - 2\sqrt{3}y = 3+\sqrt{3}$

and $x^2 + 2\sqrt{3}y = 3-\sqrt{3}$.

10. Sol. (A, D)

$$S_n < \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{1+k/n+(k/n)^2}$$

$$=\int_0^1 \frac{dx}{1+x+x^2} = \frac{\pi}{3\sqrt{3}}$$

Now, $T_n > \dfrac{\pi}{3\sqrt{3}}$ as $h\sum_{k=0}^{n-1} f(kh) > \int_0^1 f(x)\,dx > h\sum_{k=1}^{n} f(kh)$

11. Sol. (B)

For unique solution $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ where $a,b,c,d \in \{0,1\}$

Total cases $=16$.

Favorable cases $=6$ (Either $ad=1, bc=0$ or $ad=0, bc=1$).

Probability that system of equations has unique solution is $\dfrac{6}{16}=\dfrac{3}{8}$ and system of equations has either unique solution or infinite solutions so that probability for system to have a solution is 1.

12. Sol. (A)

$$D=\begin{vmatrix}1 & -2 & 3\\ -1 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}=0$$

and $D_1=\begin{vmatrix}-1 & -2 & 3\\ k & 1 & -2\\ 1 & -3 & 4\end{vmatrix}=(3-k)=0,\text{if } k=3$

$$D_2=\begin{vmatrix}1 & -1 & 3\\ -1 & k & -2\\ 1 & 1 & 4\end{vmatrix}=(k-3)=0,\text{if } k=3$$

$$D_3=\begin{vmatrix}1 & -2 & -1\\ -1 & 1 & k\\ 1 & -3 & 1\end{vmatrix}=(k-3)=0,\text{if } k=3$$

$\Rightarrow$ system of equations has no solution for $k\neq 3$.

13. Sol. (B)

$$f'(x)=g(x)\cos x+\sin x.g'(x)$$

$$\Rightarrow f'(0)=g(0)$$

$$f''(x)=2g'(x)\cos x-g(x)\sin x+\sin x\, g''(x)$$

$$\Rightarrow f''(0)=2g'(0)=0$$

But $\lim\limits_{x\to 0}\left[g(x)\cot x-g(0)\operatorname{cosec} x\right]=\lim\limits_{x\to 0}\dfrac{g(x)\cos x-g(0)}{\sin x}=\lim\limits_{x\to 0}\dfrac{g'(x)\cos x-g(x)\sin x}{\cos x}=g'(0)=0=f''(0).$

14. Sol. (D)

The direction cosines of each of the lines L_1, L_2, L_3 are proportional to $(0,1,1)$.

15. Sol. (B)

Differentiating the given equation, we get

$3y^2y'-3y'+1=0$

$\Rightarrow y'\left(-10\sqrt{2}\right)=-\frac{1}{21}$

Differentiation again we get $6yy'^2+3y^2y''-3y''=0$

$\Rightarrow f''\left(-10\sqrt{2}\right)=\frac{6.2\sqrt{2}}{(21)^4}=-\frac{4\sqrt{2}}{7^33^2}$.

16. Sol. (A)

The required area $=\int_a^b f(x)dx = xf(x)\Big|_a^b-\int_a^b xf'(x)dx$

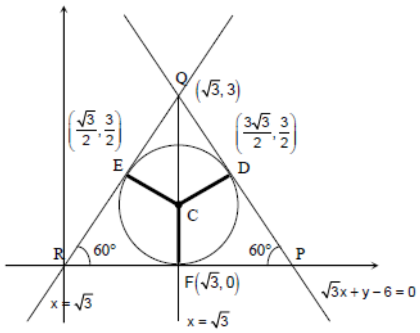
$$=bf(b)-af(a)+\int_a^b \frac{x}{3\left[\left(f(x)^2-1\right)\right]}dx.$$

17. Sol. (D)

We have $y'=\frac{1}{3\left(1-\left(f(x)^2\right)\right)}$ which is even

Hence $\int_{-1}^{1} g'(x)=g(1)-g(-1)=2g(1)$.

18. Sol. (D)



Equation of CD is $\dfrac{x-\frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}=\dfrac{y-\frac{3}{2}}{\frac{1}{2}}=-1$

$\Rightarrow C\equiv\left(\sqrt{3},1\right)$

Equation of the circle is $\left(x-\sqrt{3}\right)^2+\left(y-1\right)^2=1$.

19. Sol. (A)

Since the radius of the circle is 1 and $C\left(\sqrt{3},1\right)$, coordinates of $F\equiv\left(\sqrt{3},0\right)$

Equation of CE is $\dfrac{x-\sqrt{3}}{-\frac{\sqrt{3}}{2}}=\dfrac{y-1}{\frac{1}{2}}=1$

$\Rightarrow E\equiv\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)$.

20. Sol. (D)

Equation of QR is $y-3=\sqrt{3}\left(x-\sqrt{3}\right)$

$\Rightarrow y=\sqrt{3x}$

Equation of RP is $y=0$.

21. Sol. (B)

$A =$ Set of points on and above the line $y = 1$ in the Argand plane.

$B =$ Set of points on the circle $(x-2)^2 + (y-1)^2 = 3^2$

$C = \text{Re}(1-i)z = \text{Re}(1-i)(x+iy)$

$\Rightarrow x + y = \sqrt{2}$

Hence $(A \cap B \cap C) =$ has only one point of intersection.

22. Sol. (C)

The points $(-1,1)$ and $(5,1)$ are the extremities of a diameter of the given circle.

Hence $|z+1-i|^2 + |z-5-i|^2 = 36$

23. Sol. (D)

$||z|-|w|| < |z-w|$

and $|z-w| =$ Distance between z and w

z is fixed. Hence distance between z and w would be maximum for diametrically opposite points.

$\Rightarrow |z-w| < 6$

$\Rightarrow -6 < |z|-|w| < 6$

$\Rightarrow -3 < |z|-|w|+3 < 9.$