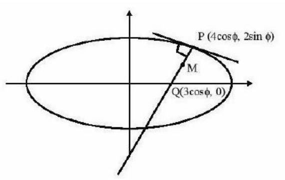
IIT-JEE-2009

PAPER– II

MATHEMATICS

SECTION – I

20. Sol (C)



Normal is $4x\sec\phi - 2y\operatorname{cose}\phi = 12$

$Q \equiv (3\cos\phi, 0)$

$M \equiv (\alpha, \beta)$

$\alpha = \dfrac{3\cos\phi + 4\cos\phi}{2} = \dfrac{7}{2}\cos\phi$

$\Rightarrow \cos\phi = \dfrac{2}{7}\alpha$

$\beta = \sin\phi$

$\cos^2\phi + \sin^2\phi = 1$

$\Rightarrow \dfrac{4}{49}\alpha^2 + \beta^2 = 1 \Rightarrow \dfrac{4}{49}x^2 + y^2 = 1$

$\Rightarrow \text{latus rectum } x = \pm 2\sqrt{3}$

$\dfrac{48}{49} + y^2 = 1 \Rightarrow y = \pm\dfrac{1}{7}$

$\left(\pm 2\sqrt{3}, \pm 1/7\right)$

21. Sol. (D)

Intersection point of $y=0$ with first line is $B(-p,0)$ Intersection point of $y-0$ with second line is $\wedge(-q,0)$ Intersection point of the two lines is $C\left(pq,(p+1)(q+1)\right)$)

Altitude from C to AB is $x=pq$

Altitude from B to AC is $y=-\frac{q}{1+q}(x+p)$

Solving these two we get $x=pq$ and $y=-pq$..

locus of orthocentre is $x+y=0$.

22. Sol. (C)

DC of the line are $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$

Any point on the line at a distance t from $P(2,-1,2)$ is $2+\frac{t}{\sqrt{3}},-1+\frac{t}{\sqrt{3}},2+\frac{t}{\sqrt{3}}$ which lies on $2x+y+z=9 \Rightarrow t=\sqrt{3}$.

23. Sol. (C)

$t_n=c\left\{n^2-(n-1)^2\right\}$

$=c(2n-1)$

$\Rightarrow t_n^2=c^2\left(4n^2-4n+1\right)$

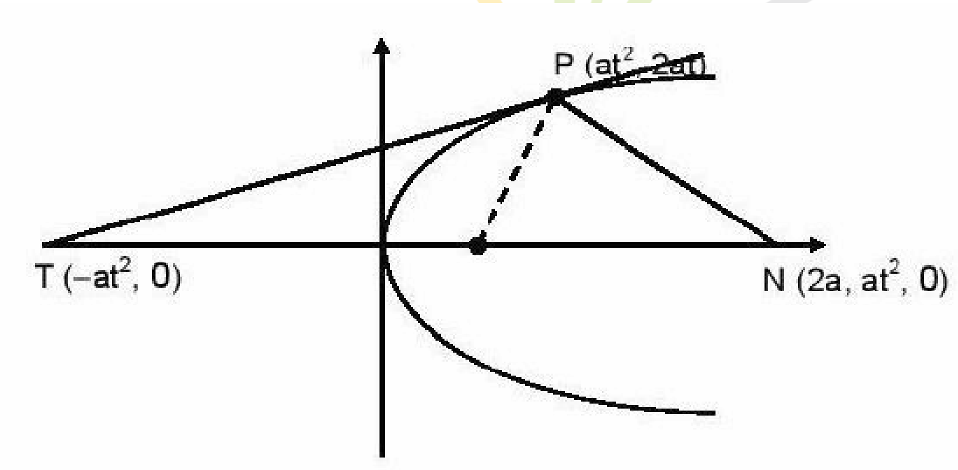
$$\sum_{n=1}^{n} t_n^2 = c^2\left\{\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right\}$$

$$= \frac{c^2 n}{6}\{4(n+1)(2n+1) - 12(n+1) + 6\}$$

$$= \frac{c^2 n}{3}\{4n^2 + 6n + 2 - 6n - 6 + 3\} = \frac{c^2}{3}n(4n^2 - 1).$$

SECTION-II

24. Sol. (A, D)



$G \equiv (h, k)$

$\Rightarrow h = \frac{2a + at^2}{3}, k = \frac{2at}{3}$

$\Rightarrow \left(\frac{3h - 2a}{a}\right) = \frac{9k^2}{4a^2}$

$\Rightarrow$ required parabola is

$$\frac{9y^2}{4a^2}=\frac{(3x-2a)}{a}=\frac{3}{a}\left(x-\frac{2a}{3}\right)$$

$$y^2=\frac{4a}{3}\left(x-\frac{2a}{3}\right)$$

$$\text{Vertex}\equiv\left(\frac{2a}{3},0\right);\text{Focus}\equiv(a,0)$$

25. Sol. (B,C,D)

For $f(x)=x\cos\left(\frac{1}{x}\right), x\geq 1$

$$f'(x)=\cos\left(\frac{1}{x}\right)+\frac{1}{x}\sin\left(\frac{1}{x}\right)\to 1 \text{ for } x\to\infty$$

also $f'(x)=\frac{1}{x^2}\sin\left(\frac{1}{x}\right)-\frac{1}{x^2}\sin\left(\frac{1}{x}\right)\frac{1}{x^3}\cos\left(\frac{1}{x}\right)$

$$--\frac{1}{x^3}\cos\left(\frac{1}{x}\right)<0 \text{ for } x\geq 1$$

$\Rightarrow f'(x)$ is decreasing for $[1,8)$

$$\Rightarrow f(x+2)<f'(x). \text{Also}, \lim_{x\to\infty} f(x+2)-f(x)=\lim_{x\to\infty}\left[(x+2)\cos\frac{1}{x+2}-x\cos\frac{1}{x}\right]=2$$

$$\therefore f(x+2)-\phi(x)>2\,\forall x\geq 1$$

26. Sol. (C,D)

Given solutions

$$\frac{1}{\sin(\pi/4)}\left[\frac{\sin(\theta+\pi/4-\theta)}{\sin\theta.\sin(\theta+\pi/4)}+\frac{\sin(\theta+\pi/2-(\theta+\pi/4))}{\sin(\theta+\pi/4).\sin\sin(\theta+\pi/2)}+L\ +\frac{\sin((\theta+3\pi/2)-(\theta+5\pi/4))}{\sin(\theta+3\pi/2).\sin\sin(\theta+5\pi/4)}\right]=4\sqrt{2}$$

$$\Rightarrow\sqrt{2}\left[\cot\theta-\cot(\theta+\pi/4)+\cot(\theta+\pi/4)-\cot(\theta+\pi/2)+K\ +\cot(\theta+5\pi/4)-\cot(\theta+3\pi/2)\right]=4\sqrt{2}$$

$$\Rightarrow\tan\theta+\cot\theta=4\Rightarrow\tan\theta=2\pm\sqrt{3}$$

$$\Rightarrow\theta=\frac{\pi}{12}\text{ or }\frac{5\pi}{12}.$$

27. Sol (A.B)

Ellipse and hyperbola will be confocal

$$\Rightarrow(\pm ae,0)=(\pm1,0)$$

$$\Rightarrow\left(\pm a\times\frac{1}{\sqrt{2}},0\right)=(\pm1,0)$$

$$\Rightarrow a=\sqrt{2}\text{ and }e=\frac{1}{\sqrt{2}}$$

$$\Rightarrow b^2=a^2(1-e^2)\Rightarrow b^2=1$$

$$\therefore\text{Equation of ellipse }\frac{x^2}{2}+\frac{y^2}{1}=1.$$

28. Sol. (A, B, C)

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x} dx$$

$$= \int_0^{\pi} \left(\frac{\sin nx}{(1+\pi^x)\sin x} + \frac{\pi^x \sin nx}{(1+\pi^x)\sin x} \right) dx = \int_0^{\pi} \frac{\sin nx}{\sin x}$$

Now $I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin nx} dx$

$$= \int_0^{\pi} \frac{2\cos(n+1)x \cdot \sin x}{\sin nx} dx = 0$$

$$\Rightarrow I_1 = \pi, -I_2 = \int_0^{\pi} 2vos\, x\, dx = 0$$

SECTION – III

29. Sol. (A) $\rightarrow$ (p) (B) $\rightarrow$ (q, s) (C) $\rightarrow$ (q, r, s, t) (D) $\rightarrow$ (r)

(A) $f'(x) > 0, \forall x \in (0, \pi/2)$

$f(0) < 0$, and $f(\pi/2) > 0$

so one solution.

(B) Let (a, b, c) is direction ratio of the intersected line, then

$ak + 4b + c = 0$

$4a + kb + 2c = 0$

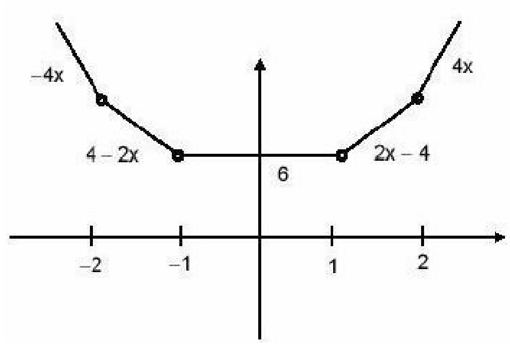
$$\frac{a}{8-k}=\frac{b}{4-2k}=\frac{c}{k^2-16}$$

We must have $2(8-k)+2(4-2k)+(k^2-16)=0$

$\Rightarrow k=2,4.$

(C) Let $f(x)=|x+2|+|x+1|+|x-1|+|x-2|$

$\Rightarrow k$ can take value 2,3,4,5.



$$\int \frac{dy}{y+1}\int dx$$

(D) $\Rightarrow f(x)=2e^x-1$

$\Rightarrow f(\ln 2)=3$

30. Sol. (A) $\to$ (q, s) (B) $\to$ (p, r, s, t) (C) $\to$ (t) (D) $\to$ (r)

(A) $2\sin^2\theta + 4\sin^2\theta\cos^2\theta = 2$

$\sin^2\theta + 2\sin^2\theta\left(1-\sin^2\theta\right) = 1$

$3\sin^2\theta - 2\sin^4\theta - 1 = 0 \Rightarrow \sin\theta \pm \frac{1}{\sqrt{2}}, \pm 1$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{\pi}{2}.$

(B). Let $y = \frac{3x}{\pi}$

$\Rightarrow \frac{1}{2} \le y \le 3 \forall x \in \left[\frac{\pi}{6}, \pi\right]$

Now $f(y) = [2y]\cos[y]$

Critical points are $y = \frac{1}{2}, y = 1, y = \frac{3}{2}, y = 3$

$\Rightarrow$ points of discontinuity $\left\{\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \pi\right\}.$

(C) $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix} = \pi \Rightarrow$ volume of parallelopiped $= \pi$

(D) $\left|\vec{a} + \vec{b}\right| = \sqrt{3}$

$\Rightarrow \sqrt{2 + 2\cos\alpha} = \sqrt{3}$

$\Rightarrow 2 + 2\cos\alpha = 3$

$\Rightarrow \alpha = \frac{\pi}{3}.$

SECTION-IV

31. Sol. 0

$$f(x)=\int_0^x f(t)dt \Rightarrow f(0)=0$$

also, $f'(x)\,f(x), x>0$

$\Rightarrow f(x)=ke^x, x>0$

Q $f(0)=0$ and $f(x)$ is continuous $\Rightarrow f(x)=0\,\forall x>0$

$\therefore f(\ln 5)=0.$

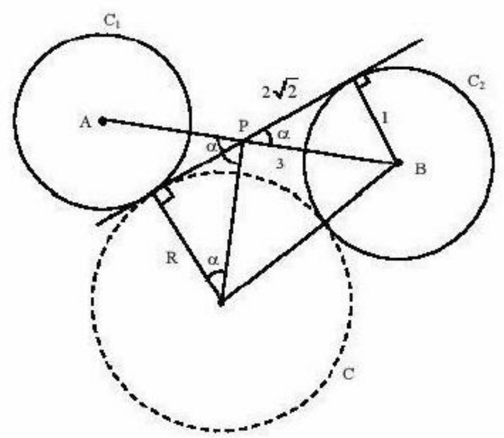
32. Sol. 8

$$\cos\alpha=\frac{2\sqrt{2}}{3}$$

$$\sin\alpha=\frac{1}{3}$$

$$\tan\alpha=\frac{2\sqrt{2}}{R}$$

$$\Rightarrow R=\frac{2\sqrt{2}}{\tan\alpha}=8\,\text{units.}$$



Alternate:

$$\left(R+1\right)^2=\left(R-1\right)^2+\left(4\sqrt{2}\right)^2$$

$$\Rightarrow R=8.$$

33. Sol. 2

$$x^2-8kx+16\left(k^2-k+1\right)=0$$

$$D>0\Rightarrow k>1 \quad \text{K } (1)$$

$$\frac{-b}{2a}>4\Rightarrow\frac{8k}{2}>4$$

$$\Rightarrow k>1 \quad \text{K } (2)$$

$$f(4)\geq 0\Rightarrow 16-2k+16\left(k^2-k+1\right)\geq 0$$

$$k^2-3k+2\geq 0$$

$$k\leq 1\cup k\geq 2 \qquad \text{K } (3)$$

Using (1), (2(, (3)

$$k_{\min}=2.$$

34. Sol.7

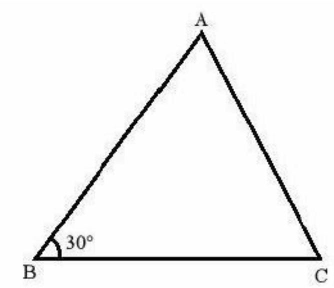
$f'(x) = 6(x-2)(x-3)$

so $f(x)$ is increasing in $(3, \infty)$

Also, $A = \{4 \le x \le 5\}$

$\therefore f_{\max} = f(5) = 7.$

35. Sol. 4



$\cos\beta = \dfrac{a^2 + 16 - 8}{2 \times a \times 4}$

$\Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{a^2 + 8}{8a}$

$\Rightarrow a^2 - 4\sqrt{3}a + 8 = 0$

$\Rightarrow a_1 + a_2 = 4\sqrt{3}, a_1, a_2 = 8$

$\Rightarrow |a_1 - a_2| = 4$

$\Rightarrow |\Delta_1 - \Delta_2| = \dfrac{1}{2} \times 4 \sin 30° \times 4 = 4.$

36. Sol. 2

$$f(0)=1,\ f'(x)=3x^2+\frac{1}{2}e^{x/2}$$

$$\Rightarrow f'(g(x))g'(x)=1$$

Put $x=0 \Rightarrow g'(1)=\dfrac{1}{f'(0)}=2.$

37. Sol. 0

Let $P(x)=ax^4+bx^3+cx^2+dx+e$

$$P'(1)=P'(2)=0$$

$$\lim_{x\to 0}\left(\frac{x^2+p(x)}{x^2}\right)=2$$

$$\Rightarrow P(0)=0 \Rightarrow d=0$$

$$\lim_{x\to 0}\left(\frac{2+p''(x)}{2}\right)=2$$

$$\Rightarrow c=1$$

On solving, $a=1/4, b=-1$

So $P(x)=\dfrac{x^4}{4}-x^3+x^2$

$$\Rightarrow P(2)=0.$$

38. Sol. 7

$$3x - y - z = 0$$
$$-3x + z = 0$$
$$\Rightarrow y = 0$$

and $z = 3x$

$$\Rightarrow x^2 + y^2 + z^2 = x^2 + z^2 = x^2 + 9x^2 = 10x^2 \le 100$$

$$\Rightarrow x^2 \le 10$$

$$\Rightarrow x = 0, \pm 1, \pm 2, \pm 3$$

There are such seven points.