JEE MAIN-2010

MATHEMATICS

29. **Sol. (C)**

$r_1, r_2, r_3 \in \{1, 2, 3, 4, 5, 6\}$

r_1, r_2, r_3 are of the form $3k, 3k+1, 3k+2$

Required probability $= \dfrac{3! \times {}^2C_1 \times {}^2C_1 \times {}^2C_1}{6 \times 6 \times 6} = \dfrac{6 \times 8}{216} = \dfrac{2}{9}$.

30. **Sol. (A)**

Evaluating midpoint of PR and QS which

gives $M \equiv \left[\dfrac{\hat{i}}{2} + \hat{j}\right]$, same for both.

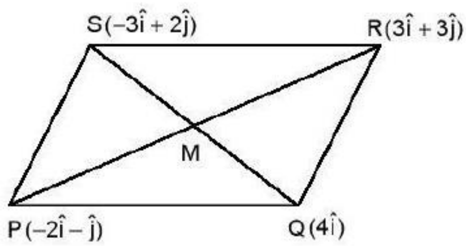
$\overrightarrow{PQ} = \overrightarrow{SR} = 6\hat{i} + \hat{j}$

$\overrightarrow{PS} = \overrightarrow{QR} = -\hat{i} + 3\hat{j}$

$\Rightarrow \overrightarrow{PQ} \cdot \overrightarrow{PS} \neq 0$

$\overrightarrow{PQ} \parallel \overrightarrow{SR}, \overrightarrow{PS} \parallel \overrightarrow{QR}$ and $|\overrightarrow{PQ}| = |\overrightarrow{SR}|, |\overrightarrow{PS}| = |\overrightarrow{QR}|$

Hence, $PQRS$ is a parallelogram but not rhombus or rectangle.



31. **Sol. (A)**

Three planes cannot intersect at two distinct points.

32. **Sol. (B)**

$$\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\,\text{In}(1+t)}{t^4+4}dt=\lim_{x\to 0}\frac{x\,\text{In}(1+x)}{(x^4+4)3x^2}$$

$$=\lim_{x\to 0}\frac{1}{3}\frac{\text{In}(1+x)}{x(x^4+4)}=\frac{1}{12}.$$

33. **Sol. (B)**

$$\alpha^3+\beta^3=q$$

$$\Rightarrow(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=q$$

$$\Rightarrow -p^3+3p\alpha\beta=q\Rightarrow \alpha\beta=\frac{q+p^3}{3p}$$

$$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=0$$

$$x^2-\frac{(\alpha^2+\beta^2)}{\alpha\beta}x+1=0$$

$$\Rightarrow x^2-\left(\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}\right)x+1=0$$

$$\Rightarrow x^2 - \frac{p^2 - 2\left(\frac{p^3+q}{3p}\right)}{\frac{p^3+q}{3p}}x + 1 = 0$$

$$\Rightarrow \left(p^3+q\right)x^2 - \left(3p^3 - 2p^3 - 2q\right)x + \left(p^3+q\right) = 0$$

$$\Rightarrow \left(p^3+q\right)x^2 - \left(p^3 - 2q\right)x + \left(p^3+q\right) = 0.$$

34. **Sol. (D)**

$$f(x) = e^{x^2} + e^{-x^2} \Rightarrow f'(x) = 2x\left(e^{x^2} - e^{-x^2}\right) \geq 0 \, \forall x \in [0,1]$$

Clearly for $0 \leq x \leq 1 \quad f(x) \geq g(x) \geq h(x)$

$\because f(1) = g(1) = h(1) = e + \frac{1}{e}$ and $f(1)$ is the greatest

$$\therefore a = b = c = e + \frac{1}{e} \Rightarrow a = b = c.$$

35. **Sol. (D)**

$$B = 60^\circ$$

$$\therefore \frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = 2\sin A + \cos C + 2\sin C\cos A$$

$$= 2\sin(A+C) = 2\sin B = 2\times\frac{\sqrt{3}}{2} = \sqrt{3}.$$

36. **Sol. (C)**

Plane 1: $ax+by+cz=0$ contains line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$

$\therefore 2a+3b+4c=0 \quad \ldots(1)$

Plane 2: $a'x+b'y+c'z=0$ is perpedicular to plane containing lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$

$\therefore 3a'+4b'+2c'=0$ and $4a'+2b'+3c'=0$

$\Rightarrow \quad \frac{a'}{12-4}=\frac{b'}{8-9}=\frac{c'}{6-16}$

$\Rightarrow \quad 8a-b-10c=0 \quad \ldots(\text{ii})$

From (i) and (ii)

$\frac{a}{-30+4}=\frac{b}{32+20}=\frac{c}{-2-24}$

$\Rightarrow$ Equation of plane $x-2y+z=0$.

37. **Sol. (A), (C), (D)**

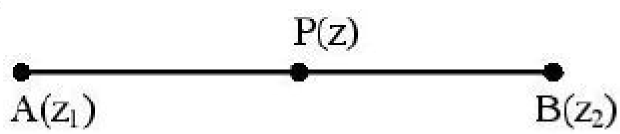
Given $z=(1-t)z_1+tz_2$

$\Rightarrow \frac{z-z_1}{z_2-z_1}=t \Rightarrow \arg\left(\frac{z-z_1}{z_2-z_1}\right)=0 \quad \ldots(1)$

$\Rightarrow \arg(z-z_1)=\arg(z_2-z_1)$

$$\frac{z-z_1}{z_2-z_1}=\frac{\overline{z}-\overline{z}_1}{\overline{z}_2-\overline{z}_1}$$

$$\begin{vmatrix} z-z_1 & \overline{z}-\overline{z}_1 \\ z_2-z_1 & \overline{z}_2-\overline{z}_1 \end{vmatrix}=0$$



$$AP+PB=AB$$

$$\Rightarrow |z-z_1|+|z-z_2|=|z_1-z_2|.$$

38. **Sol. (A)**

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$$

$$=\int_0^1\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx$$

$$=\left[\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4x^3}{3}+4x\right]_0^1-\pi$$

$$=\frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-\pi=\frac{22}{7}-\pi$$

39. **Sol. (B)**

Using cosine rule for $\angle C$

$$\frac{\sqrt{3}}{2}=\frac{\left(x^2+x+1\right)^2+\left(x^2-1\right)^2-\left(2x+1\right)^2}{2\left(x^2+x+1\right)\left(x^2-1\right)}$$

$$\Rightarrow \sqrt{3}=\frac{2x^2+2x-1}{x^2+x+1}$$

$$\Rightarrow \left(\sqrt{3}-2\right)x^2+\left(\sqrt{3}-2\right)x+\left(\sqrt{3}+1\right)=0$$

$$\Rightarrow x=\frac{\left(2-\sqrt{3}\right)\pm\sqrt{3}}{2\left(\sqrt{3}-2\right)}$$

$$\Rightarrow x=-2\left(2+\sqrt{3}\right),1+\sqrt{3}\Rightarrow x=1+\sqrt{3}\,\text{as}\left(x>0\right).$$

40. **Sol. (C), (D)**

$$A=\left(t_1^2,2t_1\right),B\left(t_2^2,2t_2\right)$$

$$\text{Centre}=\left[\frac{t_1^2+t_2^2}{2},\left(t_1+t_2\right)\right]$$

$$t_1+t_2=\pm r$$

$$m=\frac{2\left(t_1-t_2\right)}{t_1^2+t_2^2}=\frac{2}{t_1+t_2}=\pm\frac{2}{r}.$$

41. **Sol. (B), (C)**

$$f'(x)=\frac{1}{x}+\sqrt{1+\sin x}$$

$f'(x)$ is not differentiable at $\sin x=-1$ or $x=2n\pi-\frac{\pi}{2}, n\in N$

In $x\in(1,\infty)$ $f(x)>0,\ f'(x)>0$

Consider $f(x)-f'(x)$

$$=\text{In}\,x+\int_0^x\sqrt{1+\sin t}\,dt-\frac{1}{x}-\sqrt{1+\sin x}$$

$$=\left(\int_0^x\sqrt{1+\sin t}\,dt-\sqrt{1+\sin x}\right)+\text{In}\,x-\frac{1}{x}$$

Consider $g(x)=\int_0^x\sqrt{1+\sin t}\,dt-\sqrt{1+\sin x}$

It can be proved that $g(x)\geq 2\sqrt{2}-\sqrt{10}\ \forall x\in(0,\infty)$

Now there exists some $\alpha>1$ such that $\frac{1}{x}-\text{In}\,x\leq 2\sqrt{2}-\sqrt{10}$ for all $x\in(\alpha,\infty)$ as $\frac{1}{x}-\text{In}\,x$ is strictly decreasing function.

$$\Rightarrow g(x)\geq\frac{1}{x}-\text{In}\,x.$$

42. **Sol. (B)**

A tangent to $\frac{x^2}{9}-\frac{y^2}{4}=1$ is $y=mx+\sqrt{9m^2-4}, m>0$

It is tangent to $x^2+y^2-8x=0$

$\therefore \frac{4m+\sqrt{9m^2-4}}{\sqrt{1+m^2}}=4$

$\Rightarrow 495m^4+104m^2-400=0$

$\Rightarrow m^2=\frac{4}{5}$ or $m=\frac{2}{\sqrt{5}}$

$\therefore$ the tangent is $y=\frac{2}{\sqrt{5}}m+\frac{4}{\sqrt{5}}$

$\Rightarrow 2x-\sqrt{5}y+4=0.$

43. **Sol. (A)**

A point on hyperbola is $(3\sec\theta, 2\tan\theta)$

It lies on the circle, so $9\sec^2\theta+4\tan^2\theta-24\sec\theta=0$

$\Rightarrow 13\sec^2\theta-24\sec\theta-4=0 \Rightarrow \sec\theta=2,-\frac{2}{13}$

$\therefore \sec\theta=2 \Rightarrow \tan\theta=\sqrt{3}.$

The point of intersection are $A(6,2\sqrt{3})$ and $B(6,-2\sqrt{3})$

$\therefore$ The circle with AB as diameter is

$(x-6)^2+y^2=(2\sqrt{3})^2 \Rightarrow x^2+y^2-12x+24=0$.

44. **Sol. (D)**

We must have $a^2 - b^2 = kp$

$\Rightarrow (a+b)(a-b) = kp$

$\Rightarrow$ either $a-b=0$ or $a+b$ is a multiple of p

when $a=b$; number of matrices is p

and when $a + b =$ multiple of $p \Rightarrow a, b$ has $p-1$

$\therefore$ Total number of matrices $= p + p - 1$

$= 2p - 1.$

45. **Sol. (C)**

46. **Sol. (D)**

47. **Sol. (3)**

$$S_k = \frac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \frac{1}{(k-1)!}$$

$$\sum_{k=2}^{100}\left|(k^2-3k+1)\frac{1}{(k-1)!}\right|$$

$$= \sum_{k=2}^{100}\left|\frac{(k-1)^2 - k}{(k-1)!}\right|$$

$$= \sum \left| \frac{k-1}{(k-2)!} - \frac{k}{(k-1)!} \right|$$

$$= \left| \frac{2}{1!} - \frac{3}{2!} \right| + \left| \frac{3}{2!} - \frac{4}{3!} \right| + \ldots$$

$$= \frac{2}{1!} - \frac{1}{0!} + \frac{2}{1!} - \frac{3}{2!} + \frac{3}{2!} - \frac{4}{3!} + \ldots + \frac{99}{98!} - \frac{100}{99!}$$

$$= 3 - \frac{100}{99!}.$$

48. **Sol. (3)**

$$(y+z)\cos 3\theta - (xyz)3\theta = 0 \ldots (1)$$

$$xyz\sin 3\theta = (2\cos 3\theta)z + (2\sin 3\theta)y \ldots (2)$$

$$\therefore (y+z)\cos 3\theta = (2\cos 3\theta)z + (2\sin 3\theta)y = (y+2z)\cos 3\theta + y\sin 3\theta$$

$$y(\cos 3\theta - 2\sin 3\theta) = z\cos 3\theta \text{ and}$$

$$y(\sin 3\theta - \cos 3\theta) = 0 \Rightarrow \sin 3\theta - \cos 3\theta \Rightarrow 0 \sin 3\theta = \cos 3\theta$$

$$\therefore 3\theta = n\pi + \pi/4$$

49. **Sol. (9)**

$$y - y_1 = m(x - x_1)$$

Put $x = 0$, to get y intercept

$$y_1 - mx_1 = x_1^3$$

$$y_1 - x_1\frac{dy}{dx} = x_1^3$$

$$x\frac{dy}{dx} - y = -x^3$$

$$\frac{dy}{dx} - \frac{y}{x} = -x^2$$

$$e^{\int -\frac{1}{x}dx} = e^{-\text{In}\, x} = \frac{1}{x}$$

$$y \times \frac{1}{x} = \int -x^2 \times \frac{1}{x}dx$$

$$\frac{y}{x} = -\int xdx \Rightarrow \frac{y}{x} = -\frac{x^2}{2} + c$$

$$\Rightarrow f(x) = -\frac{x^3}{2} + \frac{3}{2}x \therefore f(-3) = 9.$$

50. **Sol. (3)**

$$\tan\theta = \cot 5\theta$$

$$\Rightarrow \cos 6\theta = 0$$

$$4\cos^3 2\theta - 3\cos 2\theta = 0$$

$$\Rightarrow \cos 2\theta = 0 \text{ or } \pm\frac{\sqrt{3}}{2}$$

$$\sin 2\theta = \cos 4\theta$$

$$\Rightarrow 2\sin^2 2\theta + \sin 2\theta - \sin 2\theta - 1 = 0$$

$$\sin 2\theta = -1 \text{ or } \sin 2\theta = \frac{1}{2}$$

$$\cos 2\theta = 0 \text{ and } \sin 2\theta = -1$$

$$\Rightarrow 2\theta = -\frac{\pi}{2} \Rightarrow \theta = -\frac{\pi}{4}$$

$$\cos 2\theta = \pm\frac{\sqrt{3}}{2}, \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$\therefore \theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$$

51. **Sol. (2)**

$$\frac{1}{4\cos^2\theta + 1 + \frac{3}{2}\sin 2\theta}$$

$$\Rightarrow \frac{1}{2[1+\cos 2\theta] + 1 + \frac{3}{2}\sin 2\theta}$$

lies between $\frac{1}{2}$ to $\frac{11}{2}$

$\therefore$ maximum value is 2.

Minimum value of $1+4\cos^2\theta+3\sin\theta\cos\theta$

$$1+\frac{4(1+\cos 2\theta)}{2}+\frac{3}{2}\sin 2\theta$$

$$=1+2+2\cos 2\theta+\frac{3}{2}\sin 2\theta$$

$$=3+2\cos 2\theta+\frac{3}{2}\sin 2\theta$$

$$\therefore=3-\sqrt{4+\frac{9}{4}}=3-\frac{5}{2}=\frac{1}{2}.$$

So maximum value of $\frac{1}{4\cos^2\theta+1+\frac{3}{2}\sin 2\theta}$ is 2.

52. **Sol. (5)**

$$E=(2\vec{a}+\vec{b})\cdot\left[2|\vec{b}|^2\,\vec{a}-2(\vec{a}\cdot\vec{b})\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}+|\vec{a}|^2\,\vec{b}\right]$$

$$\vec{a}\cdot\vec{b}=\frac{2-2}{\sqrt{70}}=0$$

$$|\vec{a}|=1$$

$$|b|=1$$

$$\vec{a}\cdot\vec{b}=0$$

$$E=(2\vec{a}+\vec{b})\cdot\left[2|\vec{b}|^2\,\vec{a}+|\vec{a}|^2\,\vec{b}\right]$$

$$=4|\vec{a}|^2|\vec{b}|^2+|\vec{a}|^2(\vec{a}\cdot\vec{b})+2|\vec{b}|^2(\vec{b}\cdot\vec{a})+|\vec{a}|^2|\vec{b}|^2$$

$$=5|\vec{a}|^2|\vec{b}|^2=5$$

53. **Sol. (2)**

Substituting $\left(\frac{a}{e},0\right)$ in $y=-2x+1$

$$0=-\frac{2a}{e}+1$$

$$\frac{2a}{e}=1$$

$$a=\frac{e}{2}$$

$$\text{Also,}1=\sqrt{a^2m^2-b^2}$$

$$1=a^2m^2-b^2$$

$$1=4a^2-b^2$$

$$1=\frac{4c^2}{4}-b^2$$

$$b^2=e^2-1.$$

$$\text{Also,}b^2=a^2\left(e^2-1\right)$$

$$\therefore a=1,e=2$$

54. **Sol. (6)**

$$21+3m+4n=0$$

$$31+4m+5n=0$$

$$\frac{1}{-1}=\frac{m}{2}=\frac{n}{-1}$$

Equation of plane will be

$$a(x-1)+b(y-2)+c(z-3)=0$$

$$-1(x-1)+2(y-2)-1(z-3)=0$$

$$-x+1+2y-4-z+3=0$$

$$-x+2y-z=0$$

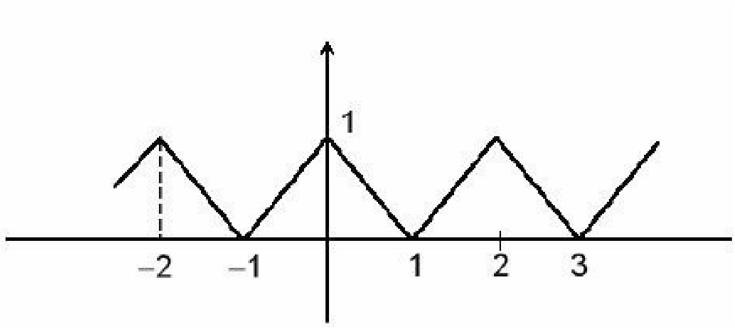
$$x-2y+z=0$$

$$\frac{|d|}{\sqrt{6}}=\sqrt{6}$$

$$d=6.$$

55. **Sol. (4)**

$$f(x)=\begin{cases} x-1, & 1\le x<2 \\ 1-x, & 0\le x<1 \end{cases}$$



$f(x)$ is periodic with period $2\sqrt{a^2+b^2}$

$$\therefore I=\int_{-10}^{10} f(x)\cos\pi x\,dx$$

$$=2\int_0^{10} f(x)\cos\pi x\,dx=2\times5\int_0^{2} f(x)\cos\pi x\,dx$$

$$=10\left[\int_0^1(1-x)\cos\pi x\,dx+\int_1^2(x-1)\cos\pi x\,dx\right]=10(I_1+I_2)$$

$$I_2=\int_1^2(x-1)\cos\pi x\,dx\,\text{put}\,x-1=t$$

$$I_2=-\int_0^1 t\cos\pi t\,dt$$

$$I_1=\int_0^1(1-x)\cos\pi x\,dx=-\int_0^1 x\cos(\pi x)\,dx$$

$$\therefore I=10\left[-2\int_0^1 x\cos\pi x\,dx\right]$$

$$=-20\left[x\frac{\sin\pi x}{\pi}+\frac{\cos\pi x}{\pi^2}\right]_0^1$$

$$=-20\left[-\frac{1}{\pi^2}-\frac{1}{\pi^2}\right]=\frac{40}{\pi^2}\therefore\frac{\pi^2}{10}I=4$$

56. **Sol. (1)**

$$\omega = e^{i2\pi/3}$$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow z\left[\left(z+\omega^2\right)\left(z+\omega\right)-1-\omega\left(z+\omega-1\right)+\omega^2\left(1-z-\omega^2\right)\right]=0$$

$$\Rightarrow z^3 = 0$$

$\Rightarrow z = 0$ is only solution.