JEE MAIN-2011

MATHEMATICS

41. **Sol.(B)**

Equation of normal at $P(6,3)$ on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{a^2x}{6}+\frac{b^2y}{3}=a^2e^2$

It intersects x-axis at $(9,0)$

$\Rightarrow a^2\frac{9}{6}=a^2e^2 \Rightarrow e=\sqrt{\frac{3}{2}}$

42. **Sol. (C)**

Let P be (h,k)

on using section formula $P\left(\frac{x}{4},\frac{y}{4}\right)$

$\therefore \quad h=\frac{x}{4}$ and $k=\frac{y}{4}$

$\Rightarrow \quad x=4h$ and $y=4k$

$\because \quad (x,y)$ lies on $y^2=4x$

$\therefore \quad 16k^2=16h \Rightarrow k^2=h$

Locus of point P is $y^2=x$.

43. **Sol. (A)**

Given $f(x)=x^2; g(x)=\sin x$

$f0g0g0f(x)=\sin^2(\sin x^2)$ and $g0g0f(x)=\sin(\sin x^2)$

given $f0g0g0f(x)=g0g0f(x) \Rightarrow \sin^2(\sin x^2)=\sin(\sin x^2)$

$\Rightarrow \sin(\sin x^2)=0$ or 1(rejected)

$\sin(\sin x^2)=0 \Rightarrow x^2=n\pi \Rightarrow x=\pm\sqrt{n\pi};\ x\in\{0,1,2,3,\ldots\}$

44. **Sol. (C)**

$$R_2=\int_{-1}^{2} f(x)dx, \quad R_1=\int_{-1}^{2} xf(x)dx$$

$$=\int_{-1}^{2}(1-x)f(1-x)dx \quad \left(\because \int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$$

$$=\int_{-1}^{2}(1-x)f(x)dx \quad (\text{given } f(x)=f(1-x))$$

$$=\int_{-1}^{2} f(x)dx-\int_{-1}^{2} xf(x)dx$$

or $R_1=R_2-R_1 \Rightarrow 2R_1=R_2$

45. **Sol. (D)**

$$\lim_{x\to 0}\left[1+x\ell n\left(1+b^2\right)\right]^{\frac{1}{x}}=e^{\lim_{x\to 0}\frac{x\ell n\left(1+b^2\right)}{x}}=1+b^2$$

Hence $1+b^2=2b\sin^2\theta$

$$\Rightarrow \quad \sin^2\theta = \frac{1}{2}\left(b+\frac{1}{b}\right) \geq 1$$

$$\therefore \quad \sin^2\theta = 1 \Rightarrow \quad \sin\theta = \pm 1 \quad \Rightarrow \theta = \pm\frac{\pi}{2}$$

46. **Sol. (D)**

Family of circle which touches y-axis at $(0,2)$ is

$$x^2 + (y-2)^2 + \lambda x = 0$$

Passing through $(-1,0)$

$$\Rightarrow \quad 1+4-\lambda = 0 \quad \Rightarrow \lambda = 5$$
$$\therefore \quad x^2 + y^2 + 5x - 4y + 4 = 0$$

which satisfy the point $(-4,0)$.

47. **Sol. (A)**

$$\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$$

$$= 1 - c\omega - a\left(\omega - \omega^2 c\right) = \quad (1-c\omega) - a\omega(1-c\omega) = (1-c\omega)(1-a\omega)$$

for non singular matrix

$$c \neq \frac{1}{\omega} \;\&\; a \neq \frac{1}{\omega}$$

$$\Rightarrow c \neq \omega^2, \quad a \neq \omega^2$$

$\Rightarrow a \,\&\, c$ must be ω & b can be ω or ω^2

$\therefore$ total matrices $= 2$

48. **Sol. (B)**

$$\frac{x^2}{b^2+1} = \frac{-x}{b+1} = \frac{1}{1-b}$$

$$\Rightarrow \quad x = \frac{b+1}{b-1} \quad \ldots(\text{i})$$

$$\& \quad x^2 = \frac{b^1+1}{1-b} \quad \ldots(\text{ii})$$

From (i) & (ii)

$$\left(\frac{b+1}{b-1}\right)^2 = \frac{b^2+1}{1-b}$$

$$\Rightarrow (b^2+1)(1-b) = (b+1)^2 \quad \Rightarrow -b^3+1+b^2-b = b^2+1+2b$$

$$\Rightarrow -b^3-3b = 0 \quad \Rightarrow b(b^2+3) = 0$$

$$\Rightarrow b = 0, \quad b = \pm\sqrt{3}\,i$$

49. **Sol. (A,B,C,D)**

$$f\left(-\frac{\pi^-}{2}\right) = 0,\ f\left(-\frac{\pi^+}{2}\right) = 0$$

$$f'(x) = \begin{cases} -1 & x \le \frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} < x \le 0 \\ 1 & 0 < x \le 1 \\ \frac{1}{x} & x > 1 \end{cases}$$

$f'(0^-)=0, f'(0^+)=1$ $\therefore$ not differentiable at $x=0$

$f'(1^-)=1, f'(1^+)=1$ $\therefore$ not differentiable at $x=1$

as $-\frac{3}{2}\in\left(-\frac{\pi}{2},0\right)$

$f'(x)=\sin x$ which is differentiable at $x=-\frac{3}{2}$

50. **Sol. (A,B,D)**

Equation of normal is $y=mx-2m-m^3$

It passes through the point $(9,6)$ then

$6=9m-2m-m^3$

$\Rightarrow m^3-7m+6=0\Rightarrow (m-1)(m-2)(m+3)=0\Rightarrow m=1,2,-3$

Equations of normals are $y-x+3=0, y+3x-33=0$ & $y-x+12=0$

51. **Sol. (A,D)**

Let $P(E)=x$ & $P(F)=y$

According to given condition

$x(1-y)+y(1-x)=\frac{11}{25}$

$\Rightarrow x+y-2xy=\frac{11}{25}$ $\ldots$(i)

Also, $(1-x)(1-y)=\frac{2}{5}$

$\Rightarrow x+y-xy=\frac{23}{25} \quad \ldots(\text{ii})$

from (i) & (ii)

$xy=\frac{12}{24}, x+y=\frac{7}{5}$

Solving this $x=\frac{4}{5}, y=\frac{3}{5}$ or $x=\frac{3}{5}, y=\frac{4}{5}$

52. **Sol. (A)**

$f;(0,1)\rightarrow R$

$f(x)=\frac{b-x}{1-bx} \quad b\in(0,1)$

$\Rightarrow \quad f'(x)=\frac{b^2-1}{(bx-1)^2}$

$\Rightarrow \quad f'(x)<0 \forall x\in(0,1)$

hence $f(x)$ is decreasing function

hence its range $(-1,b)$

$\Rightarrow$ co-domain $\neq$ range

$\Rightarrow \quad f(x)$ is non-invertible function

53. **Sol. (0)**

Given $y(0)=0, g(0)=g(2)=0$

Let $y'(x)+y(x)\cdot g'(x)=g(x)g'(x) \quad \Rightarrow y'(x)+(y(x)-g(x))g'(x)=0$

$$\Rightarrow \frac{y'(x)}{g'(x)}+y(x)=g(x) \quad \Rightarrow \frac{dy(x)}{dg(x)}+y(x)=g(x)$$

$$\Rightarrow I.F.=e^{\int d(g(x))}=e^{g(x)} \quad \Rightarrow y(x).e^{g(x)}=\int e^{g(x)}g(x).dg(x)$$

$y(x).e^{g(x)}=g(x).e^{g(x)}-e^{g(x)}+c$

put $x=0$

$\Rightarrow 0=0-1+c \quad \Rightarrow c=1$

$\Rightarrow y(2)\cdot e^{g(2)}=g(2)e^{g(2)}-e^{g(2)}+1$

$\Rightarrow y(2)=0-e^{0}+1 \quad \Rightarrow y(2)=0$

54. **Sol. (9)**

$\vec{a}=-\hat{i}-\hat{k}$

$\vec{b}=-\hat{i}+\hat{j}$

$\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$

Taking crass product by $\vec{a}$

$(\vec{r}\times\vec{b})\times\vec{a}=(\vec{c}\times\vec{b})\times\vec{a}$

$\Rightarrow (\vec{r}.\vec{a})\vec{b}-(\vec{b}.\vec{a})\vec{r}=(\vec{c}.\vec{a})\vec{b})-(\vec{b}.\vec{a})\vec{c} \quad \Rightarrow 0-\vec{r}=(-1-3)(-\hat{i}+\hat{j})-(1)(\hat{i}+2\hat{j}+3\hat{k})$

$\vec{r}=-3\hat{i}+6\hat{j}+3\hat{k}$

$\vec{r}.\vec{b}=3+6=9$

55. **Sol. (3)**

Comment: If $\omega = e^{i\pi/3}$ then $\dfrac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is not always an integer, infect its value depends upon a, b, c

$\Rightarrow$ Let $\omega = e^{i2\pi/3}$

$$|x|^2 = (a+b+c)(\bar{a}+\bar{b}+\bar{c})$$
$$= |a|^2+|b|^2+|c|^2+a\bar{b}+a\bar{c}+b\bar{a}+b\bar{c}+c\bar{a}+c\bar{b}$$
$$|y|^2 = (a+b\omega+c\omega^2)(\bar{a}+\bar{b}\omega^2+\bar{c}\omega)$$
$$= |a|^2+|b|^2+|c|^2+a\bar{b}\omega^2+a\bar{c}\omega+b\bar{a}\omega+b\bar{c}\omega^2+c\bar{a}\omega^2+c\bar{b}\omega$$
$$|z|^2 = (a+b\omega^2+c\omega)(\bar{a}+\bar{b}\omega+\bar{c}\omega^2)$$
$$= |a|^2+|b|^2+|c|^2+a\bar{b}\omega+a\bar{c}\omega^2+b\bar{a}\omega^2+b\bar{c}\omega+c\bar{a}\omega+c\bar{b}\omega^2$$
$$\therefore |x|^2+|y|^2+|z|^2 = 3\left(|a|^2+|b|^2+|c|^2\right)$$
$$\Rightarrow \frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2} = 3$$

56. **Sol. (9)**

$$\text{Let } M = \begin{bmatrix} a & b & c \\ x & y & z \\ \ell & m & n \end{bmatrix}$$

according to question

$$\begin{bmatrix} a & b & c \\ x & y & z \\ \ell & m & n \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$
$$\Rightarrow \quad b=-1, y=2, m=3 \quad \ldots(1)$$

$$\Rightarrow \begin{bmatrix} a & b & c \\ x & y & z \\ \ell & m & n \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$a - b = 1$

$x - y = 1$

$\ell - m = -1$

form (1) $a = 0$

$x = 3$

$\ell = 2$

$$\begin{bmatrix} a & b & c \\ x & y & z \\ \ell & m & n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

$\ell + m + n = 12$

$\Rightarrow 2 + 3 + n = 12 \Rightarrow n = 7$

Now $a + y + n = 0 + 2 + 7 = 9$

57. **Sol. (2)**

Let

$$\begin{aligned} f(x) &= x^4 - 4x^3 + 12x^2 + x - 1 \\ f'(x) &= 4x^3 - 12x^2 + 24x \\ f''(x) &= 12x^2 - 24x + 24 \\ &= 12\left(x^2 - 2x + 2\right) > 0 \end{aligned}$$

$\Rightarrow f'(x)$ is strictly increasing function

$\because f'(x)$ is cubic polynomial

hence number of loots of $f'(x)=0$ is 1

$\Rightarrow$ Number of maximum roots of $f(x)=0$ are 2

Now $f(0)=-1, f(1)=9, f(-1)=15$

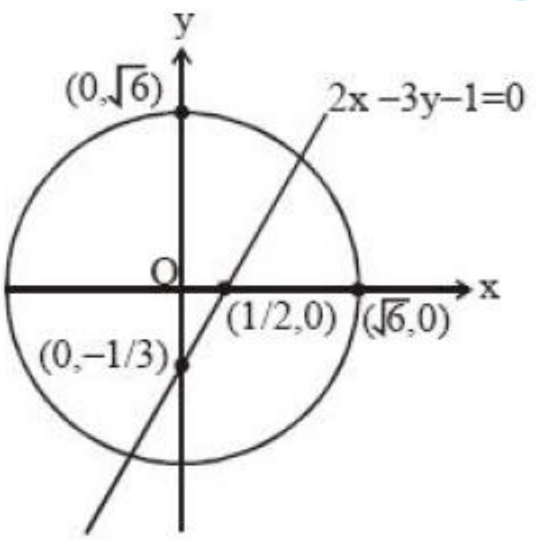
$\Rightarrow f(x)$ has exactly 2 distinct real roots.

58. **Sol. (2)**

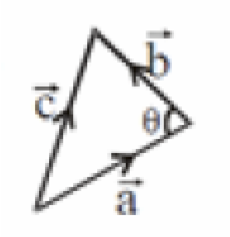
If the point lies inside the smaller part, then origin and point should give opposite signs w.r. t. line & point should lie inside die circle.

for origin : $2\times 0-3\times 0-1=-1(-ve)$ for $\left(2,\frac{3}{4}\right): 2\times 2-3\times\frac{3}{4}-1=\frac{3}{4}(+ve)$; point lies inside the circle for $\left(\frac{5}{2},\frac{3}{4}\right): 2\times\frac{5}{2}-3\times\frac{3}{4}-1=\frac{7}{4}(+ve)$; point lies outside the circle For $\left(\frac{1}{4},-\frac{1}{4}\right): 2\times\frac{1}{4}-3\left(-\frac{1}{4}\right)-1=\frac{1}{4}(+ve)$; point lies inside the circle For $\left(\frac{1}{8},\frac{1}{4}\right): 2\times\frac{1}{8}-3\left(\frac{1}{4}\right)-1=\frac{-3}{2}(-ve)$; point lies inside the circle.

$\therefore$ 2 points lie inside smaller part.



59. **Sol. (A)$\rightarrow$ (q); (B)$\rightarrow$ (p); (C)$\rightarrow$ (s); (D)$\rightarrow$ (t)**

(A) 

$|\vec{a}| = 2$

$|\vec{b}| = 2$

$|\vec{c}| = 2\sqrt{3}$

$$\cos\theta = \frac{|\vec{a}| + |\vec{b}|^2 - |\vec{c}|^2}{2|\vec{a}||\vec{b}|} = \frac{4+4-12}{2.2.2} = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

(B) $\int_a^b \left(f(x) - 3x \right) dx = a^2 - b^2 = \int_a^b (-2x) dx$

$\Rightarrow \int_a^b \left(f(x) - x \right) dx = 0 \Rightarrow$ one of the possible solution of this equation is

$$f(x) = x \Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$$

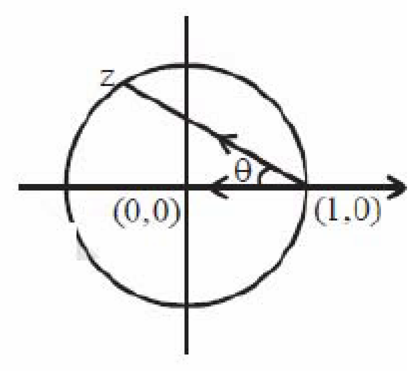
(C) $\frac{\pi^2}{\ell n 3} \int_{7/6}^{5/6} (\sec \pi x)\, dx = \frac{\pi^2}{\ell n 3} \frac{1}{\pi} \left[\ell n |\sec \pi x + \tan \pi x| \right]_{7/6}^{5/6}$

$$= \frac{\pi}{\ell n 3} \ell n \left| \frac{\sec\frac{5\pi}{6} + \tan\frac{5\pi}{6}}{\sec\frac{7\pi}{6} + \tan\frac{7\pi}{6}} \right| = \frac{\pi}{\ell n 3} \ell n 3 = \pi$$

(D) Let $\theta = Arg\left(\frac{1}{1-z}\right)$

$\Rightarrow \theta = Arg\left(\frac{1}{1-z}\right)$ which is shown in adjacent diagram.

$\Rightarrow$ Maximum value of θ is approaching to $\frac{\pi}{2}$



60. **Sol. (A)$\rightarrow$ (s); (B)$\rightarrow$ (t); (C)$\rightarrow$ (r); (D)$\rightarrow$ (r)**

(A) Let $z=\cos\theta+i\sin\theta$

$$\text{Re}\left(\frac{2i(\cos\theta+i\sin\theta)}{1-(\cos\theta+i\sin\theta)^2}\right)=\text{Re}\left(\frac{\cos\theta i-\sin\theta}{\sin^2\theta-i\cos\theta\sin\theta}\right)$$

$$=\text{Re}\left(-\frac{1}{\sin\theta}\right)=\frac{-1}{\sin\theta}$$

$\therefore$ Set will be $(-\infty,-1]\cup[1,\infty)$

(B) $-1\le\frac{8.3^{(x-2)}}{1-3^{2(x-1)}}\le 1 \quad x\ne 1$

$$\Rightarrow \quad -1\le\frac{8.3^x}{(3-3^x)(3+3^x)}\le 1$$

$3^x=1 \quad \therefore t>0$

$$\frac{8t}{(3-t)(t+3)}\ge -1\Rightarrow \quad t\in(0,3)\cup[9,\infty)\Rightarrow \quad x\in(-\infty,1)\cup[2,\infty)$$

$$\frac{8t}{(3-t)(t+3)} \le 1 \Rightarrow \quad t \in (0,1] \cup (3,\infty) \Rightarrow \quad x \in [-\infty,0) \cup (1,\infty)$$

taking intersection,

$$x \in (-\infty,0] \cup [2,\infty)$$

(C) $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$

$C_1 \to C_2 + C_3$

$$\Rightarrow f(\theta) = \begin{vmatrix} 2 & \tan\theta & 1 \\ 0 & 1 & \tan\theta \\ 0 & -\tan\theta & 1 \end{vmatrix}$$

$$\Rightarrow f(\theta) = 2\sec^2\theta \quad \Rightarrow f(\theta) \in [2,\infty)$$

(D) $f(x) = 3x^{5/2} - 10x^{3/2}$

$$f'(x) = \frac{15}{2}x^{3/2} - \frac{30}{2}x^{1/2} > 0$$

$$\Rightarrow \frac{15}{2}\sqrt{x}(x-2) \ge 0 \quad \Rightarrow x \ge 2$$