IIT-JEE-2012

PAPER-1

MATHEMATICS

41. Sol: (B)

Given

$$\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$$

$$\Rightarrow \lim_{x\to\infty}\frac{x^2+x+1-ax^2-ax-bx-b}{(x+1)}=4 \Rightarrow \lim_{x\to\infty}\frac{(1-a)x^2+(1-a-b)x+(1-b)}{(x+1)}=4$$

$\Rightarrow 1-a=0$ and $1-a-b=4 \Rightarrow b=-4, a=1.$

42. Sol: (D)

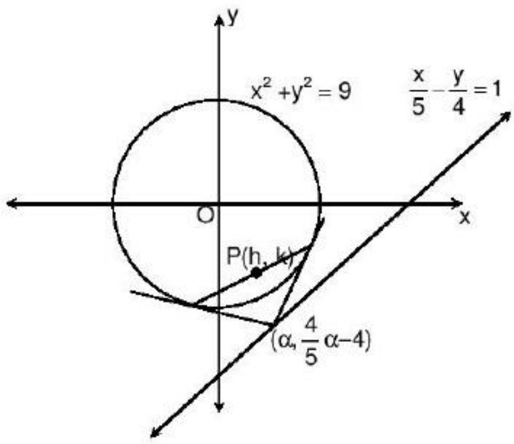
$$|Q|=\begin{vmatrix}2^2a_{11} & 2^3a_{12} & 2^4a_{13}\\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23}\\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33}\end{vmatrix} \Rightarrow |Q|=2^2\cdot2^3\cdot2^4\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ 2a_{21} & 2a_{22} & 2a_{23}\\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33}\end{vmatrix}$$

$$|Q|=2^9\cdot2\cdot2^2\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$\Rightarrow |Q|=2^{12}|P|$$

$$|Q|=2^{13}.$$

43. Sol: (A)



Equitation of the chord bisected at $P(h,k)$

$$hk+ky=h^2+k^2 \quad \text{K (i)}$$

Let any point on the line be $\left(\alpha, \frac{4}{5}\alpha-4\right)$

Equation of the chord of contact is

$$\Rightarrow \alpha x+\left(\frac{4}{5}\alpha-4\right)y=9 \quad \text{K (ii)}$$

Comparing (i) and (ii)

$$\frac{h}{\alpha}=\frac{k}{\frac{4}{5}\alpha-4}=\frac{h^2+k^2}{9}$$

$$\alpha=\frac{20h}{4h-5k}$$

Now, $\dfrac{h(4h-5k)}{20h}=\dfrac{h^2+k^2}{9}$

$$20\left(h^2+k^2\right)=9(4h-5k)$$

$$20\left(x^2+y^2\right)-36x+45y=0$$

44. Sol: (B)

Number of ways

$$= 3^5 - {}^3C_1.2^5 + {}^3C_2 1^5$$
$$= 243 - 96 + 3 = 150$$

45. Sol: (C)

$$I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$$

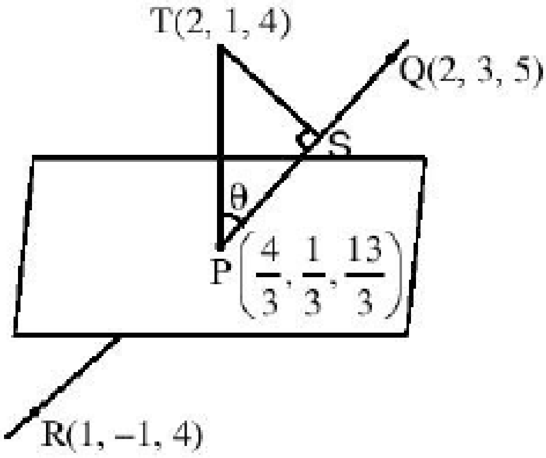
Let $\sec x + \tan x = t$

$$\Rightarrow \sec x - \tan x = 1/t$$

Now $\left(\sec x \tan x + \sec^2 x\right) dx = dt$

$$\sec x \left(\sec x + \tan x\right) dx = dt$$
$$\sec x dx = \frac{dt}{t}, \frac{1}{2}\left(t + \frac{1}{t} = \sec x\right)$$
$$I = \frac{1}{2}\int \frac{\left(t + \frac{1}{t}\right)}{t^{9/2}} \frac{dt}{t}$$
$$= \frac{1}{2}\int \left(t^{-9/2} + t^{-13/2}\right) dt$$
$$= \frac{1}{2}\left[\frac{t^{-9/2+1}}{-9/2+1} + \frac{t^{-13/2+1}}{-13/2+1}\right]$$
$$= \frac{1}{2}\left[\frac{t^{-7/2}}{-7/2} + \frac{t^{-11/2}}{-11/2}\right]$$
$$= -\frac{1}{7}t^{-7/2} - \frac{1}{11}t^{-11/2}$$
$$= -\frac{1}{7}\frac{1}{t^{7/2}} - \frac{1}{11}\frac{1}{t^{11/2}}$$
$$= \frac{1}{t^{11/2}}\left(\frac{1}{11} + \frac{t^2}{7}\right) = \frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + k$$

46. Sol: (A)



$D.R.$ of QR is $1,4,1$

$Coordinate\ of\ P \equiv \left(\frac{4}{3},\frac{1}{3},\frac{13}{3}\right)$

$D.R.$ of PT is $2,2,-1$

Angle between QR and PT is $45°$

And $PT = 1$

$\Rightarrow PS = TS = \frac{1}{\sqrt{2}}$

47. Sol: (B)

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{h}$$

$$= \lim_{h\to 0} h\cos\left(\frac{\pi}{h}\right) = 0$$

So, $f(x)$ is differentiable at $x = 0$

$$f'(2^+)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2\left|\cos\frac{\pi}{2+h}\right|-0}{h}$$

$$=\lim_{h\to 0}\frac{(2+h)^2\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$f'(2^+)=\lim_{h\to 0}\frac{(2+h)^2}{h}\sin\left(\frac{\pi}{2}-\frac{\pi}{2+h}\right)$$

$$=\lim_{h\to 0}\frac{(2+h)^2}{h}\sin\left[\frac{\pi\cdot h}{2(2+h)}\right]$$

$$=\lim_{h\to 0}\frac{(2+h)^2}{\frac{\pi h}{2(2+h)}}\sin\frac{\pi h}{2(2+h)}\times\frac{\pi}{2(2+h)}=\pi$$

Again, $f'(2^-)=\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

$$=\lim_{h\to 0}\frac{-(2-h)^2\left|\cos\left(\frac{\pi}{2-h}\right)\right|}{-h}$$

$$=\lim_{h\to 0}\frac{-(2-h)^2\cos\left(\frac{\pi}{2-h}\right)}{-h}$$

$$\lim_{h\to 0}\frac{(2-h)^2\sin\left[\frac{\pi}{2}-\frac{\pi}{2-h}\right]}{h}$$

$$\lim_{h\to 0}\frac{(2-h)^2}{h}\cdot\sin\left[\frac{-\pi h}{2(2-h)}\right]$$

$$-\lim_{h\to 0}\frac{(2-h)^2}{\frac{\pi h}{2(2-h)}}\cdot\sin\frac{\pi h}{2(2-h)}\times\frac{\pi}{2(2-h)}=-\pi$$

48. Sol: (D)

Given equation $z^2+z+1-a=0$

Clearly this equation do not have real root if

$$D<0$$
$$\Rightarrow 1-4(1-a)<0$$
$$\Rightarrow 4a<3$$
$$a<\frac{3}{4}.$$

49. Sol: (C)

Equation of ellipse is $(y+2)(y-2)+\lambda(x+3)(x-3)=0$

It passes through $(0,4)\Rightarrow \lambda=\frac{4}{5}$

Equitation of ellipse is $\frac{x^2}{12}+\frac{y^2}{16}=1$

$$e=\frac{1}{2}$$

Alternate

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as it is passing through $(0,4)$ and $(3,2)$

So, $b^2=16$ and $\frac{9}{a^2}+\frac{4}{16}=1$

$$\Rightarrow a^2=12$$

So, $12=16(1-e^2)$

$$\Rightarrow e=1/2$$

50. Sol: (B)

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$
$$f'(x) = 6x^2 - 30x + 36$$
$$= 6(x^2 - 5x + 6)$$
$$= 6(x-2)(x-3)$$

$f(x)$ is increasing in $[0,2]$ and decreasing in $[2,3]$

$f(x)$ is many one

$$f(0) = 1$$
$$f(2) = 29$$
$$f(3) = 28$$

Range is $[1, 29]$

Hence, $f(x)$ is many-one-onto

51. Sol: (A, B)

$slope\,of\ \tan gent = 2$

The tangent are $y = 2x \pm \sqrt{9 \times 4 - 4}$

$i.e., 2x - y = \pm 4\sqrt{2}$

$$\Rightarrow \frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1 \text{ and } -\frac{x}{2\sqrt{2}} + \frac{y}{4\sqrt{2}} = 1$$

Comparing it with $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$

We get point contact as $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Alternative :

Equation of the tangent at $P(\theta)$ is $\left(\frac{\sec\theta}{3}\right)x - \left(\frac{\tan\theta}{2}\right)y = 1$

$\Rightarrow Slope = \frac{2\sec\theta}{3\tan\theta} = 2$

$\Rightarrow \sin\theta = \frac{1}{3}$

$\Rightarrow$ points are $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

52. Sol: (A, C, D)

$$2\cos\theta(1-\sin\varphi) = \frac{2\sin^2\theta}{\sin\theta}\cos\varphi - 1 = 2\sin\theta\cos\varphi - 1$$
$$2\cos\theta - 2\cos\theta\sin\varphi = 2\sin\theta\cos\varphi - 1$$
$$2\cos\theta + 1 = 2\sin(\theta+\varphi)$$

$$\tan(2\pi-\theta) > 0 \Rightarrow \tan\theta < 0 \text{ and } -1 < \sin\theta < -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$$
$$\frac{1}{2} < \sin(\theta+\varphi) < 1$$
$$\Rightarrow 2\pi + \frac{\pi}{6} < \theta + \varphi < \frac{5\pi}{6} + 2\pi$$
$$2\pi + \frac{\pi}{6} < \theta + \varphi < \frac{5\pi}{6} + 2\pi$$
$$2\pi + \frac{\pi}{6} - \theta_{max} < \varphi < 2\pi + \frac{5\pi}{6} - \theta_{min}$$
$$\frac{\pi}{2} < \varphi < \frac{4\pi}{3}.$$

53. Sol: (A, D)

$$\frac{dy}{dx} - y\tan x = 2x\sec x$$

$$\cos x\frac{dy}{dx} + (-\sin x)y = 2x$$

$$\frac{d}{dx}(y\cos x) = 2x$$

$$y(x)\cos x = x^2 + c, \text{ where } c = 0 \text{ since } y(0) = 0$$

$$\text{when } x = \frac{\pi}{4}, y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}, \quad \text{when } x = \frac{\pi}{3}, y\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9}$$

$$\text{When } x = \frac{\pi}{4}, y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}} + \frac{\pi}{\sqrt{2}}$$

$$\text{When } x = \frac{\pi}{3}, y'\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{3\sqrt{3}} + \frac{4\pi}{3}$$

54. Sol: (B, D)

$$P(X_1) = \frac{1}{2}, P(X_2) = \frac{1}{4}, P(X_3) = \frac{1}{4}$$

$$P(X) = P(X_1 \cap X_2 \cap X_3^C) + P(X_1 \cap X_2^C \cap X_3) + P(X_1 \cap X_2 \cap X_3) = \frac{1}{4}$$

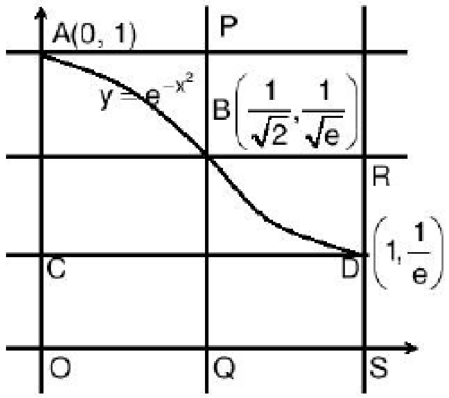
(A) $P(X_1^C / X) = \dfrac{P(X \cap X_1^C)}{P(X)} = \dfrac{\frac{1}{32}}{\frac{1}{4}} = \dfrac{1}{8}$

(B) $P[\textit{exactly two engines of the ship are functioning} \mid X] = \dfrac{\frac{7}{32}}{\frac{1}{4}} = \dfrac{7}{8}$

(C) $P\left(\dfrac{X}{X_2}\right) = \dfrac{\frac{5}{32}}{\frac{1}{4}} = \dfrac{5}{8}$

(D) $P\left(\dfrac{X}{X_1}\right) = \dfrac{\frac{7}{32}}{\frac{1}{2}} = \dfrac{7}{16}$

55. Sol: (A, B, D)



$S > \frac{1}{e}$ (As area of rectangle $OCDS = 1/e$)

Since $e^{-x^2} \geq e^{-x} \forall x \in [0,1]$

$\Rightarrow S > \int_0^1 e^{-x} dx = \left(1 - \frac{1}{e}\right)$

$Area\ of\ the\ rectangle\ OAPQ + Area\ of\ the\ rectangle\ QBRS > S$

$S < \frac{1}{\sqrt{2}}(1) + \left(1 + -\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right).$

Since $\frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right) < 1 - \frac{1}{e}$

Hence, (C) is incorrect.

56. Sol: (3)

As, $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 3\left(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2\right) - |\vec{a} + \vec{b} + \vec{c}|^2$

$\Rightarrow 3 \times 3 - |\vec{a} + \vec{b} + \vec{c}|^2 = 9$

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}| = 0 \Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$

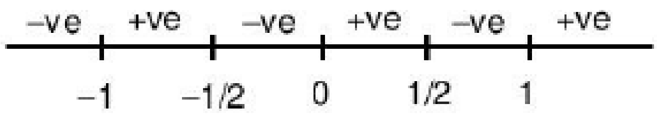
$\Rightarrow \vec{b} + \vec{c} = -\vec{a}$

$\Rightarrow \left|2\vec{a} + 5\left(\vec{b} + \vec{c}\right)\right| = |-3\vec{a}| = 3|\vec{a}| = 3$

57. Sol: (5)

$$f'(x)=\frac{|x|}{x}+\frac{|x^2-1|}{x^2-1}\cdot(2x)$$

$$=\begin{cases}2x-1 & , \quad x<-1\\ -(2x+1) & , \quad -1<x<0\\ 1-2x & , \quad 0<x<1\\ 2x+1 & , \quad x>1\end{cases}$$



So, $f'(x)$ changes sign at point

$$x=-1,-\frac{1}{2},0,\frac{1}{2},1$$

So, total number of points of local maximum or minimum is 5

58. Sol: (4)

The parabola is $x=2t^2, y=4t$

Solving it with the circle we get :

$$4t^4+16t^2-4t^2-16t=0$$

$$\Rightarrow t^4+3t^2-4t=0 \Rightarrow t=0,1$$

So, the point P and Q are $(0,0)$ and $(2,4)$ which are also diametrically opposite points on the circle. The focus is $S\equiv(2,0)$.

$$The\,area\,of\ \Delta PQS=\frac{1}{2}\times2\times4=4.$$

59. Sol: (9)

Let $p'(x)=k(x-1)(x-3)$

$$\Rightarrow p(x)=k\left(\frac{x^3}{3}-2x^2+3x\right)+c$$

Now, $p(1)=6\Rightarrow\frac{4}{3}k+c=6$

Also, $p(3)=2\Rightarrow c=2$

So, $k=3$, so, $p'(0)=3k=9.$

60. Sol: (4)

Let $\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\text{K}}}}=y$

So, $4-\frac{1}{3\sqrt{2}}y=y^2 \quad (y>0)$

$$\Rightarrow y^2+\frac{1}{3\sqrt{2}}y-4=0\Rightarrow y=\frac{8}{3\sqrt{2}}$$

So, the required value is $6+\log_{3/2}\left(\frac{1}{3\sqrt{2}}\times\frac{8}{3\sqrt{2}}\right)$

$$=6+\log_{3/2}\frac{4}{9}=6-2=4.$$