IIT-JEE-2012 PAPER 2

PART-I: Mathematics

41 Sol. (D)

$a_1, a_2, a_3,$ are in H.P.

$\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots\ldots$ are in A.P.

$\Rightarrow \frac{1}{a_n} = \frac{1}{a_1} + (n-1)d < 0,$ where $\frac{\frac{1}{25} - \frac{5}{25}}{19} = d = \left(\frac{-4}{9 \times 25}\right)$

$$\Rightarrow \frac{1}{5} + (n-1)\left(\frac{-4}{19 \times 25}\right) < 0$$

$$\frac{4(n-1)}{19 \times 5} > 1$$

$$n - 1 > \frac{19 \times 5}{4}$$

$$n > \frac{19 \times 5}{4} + 1 \Rightarrow n \geq 25.$$

42 Sol. (A)

Equation of required plane is

$$P \equiv (x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$$

$$\Rightarrow (1+\lambda)x + (2-\lambda)y + (3+\lambda)z - (2+3\lambda) = 0$$

Its distance from $(3,1,-1)$ is $\frac{2}{\sqrt{3}}$

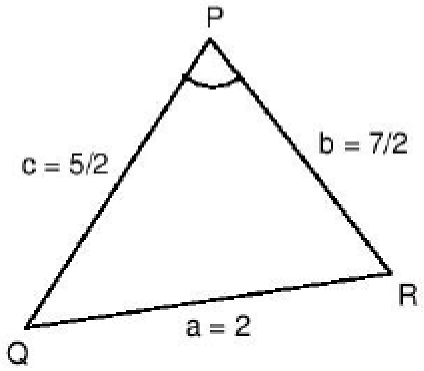
$$\Rightarrow \frac{2}{\sqrt{3}} = \frac{|3(1+\lambda) + (2-\lambda) - (3+\lambda) - (2+3\lambda)|}{\sqrt{(\lambda+1)^2 + (2-\lambda)^2 + (3+\lambda)^2}}$$

$$= \frac{4}{3} = \frac{(-2\lambda)^2}{3\lambda^2 + 4\lambda + 14} \Rightarrow 3\lambda^2 + 4\lambda + 14 = 3\lambda^2$$

$$\Rightarrow \lambda = -\frac{7}{2} \Rightarrow -\frac{5}{2}x + \frac{11}{2}y - \frac{z}{2} + \frac{17}{2} = 0$$

$$-5x + 11y - z + 17 = 0$$

43 Sol. (C)



$$\frac{2\sin P-2\sin P\cos P}{2\sin P+2\sin P\cos P}=\frac{1-\cos P}{1+\cos P}=\frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}}=\tan^2\frac{P}{2}$$

$$=\frac{(s-b)(s-c)}{s(s-a)}$$

$$=\frac{((s-b)(s-c))^2}{\Delta^2}=\frac{\left(\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\right)^2}{\Delta^2}=\left(\frac{3}{4\Delta}\right)^2$$

44 Sol. (C)

$$\vec{a}\times\left(2\hat{i}+3j+4k\right)=\left(2\hat{i}+3j+4k\right)\times\vec{b}$$

$$\left(\vec{a}+\vec{b}\right)\times\left(2\hat{i}+3j+4k\right)=\vec{0}$$

$$\Rightarrow \vec{a}+\vec{b}=\pm\left(2\hat{i}+3j+4k\right)\quad\left(as\left|\vec{a}+\vec{b}\right|=\sqrt{29}\right)$$

$$\Rightarrow\left(\vec{a}+\vec{b}\right)\cdot\left(-7\hat{i}+2j+3k\right)$$

$$=\pm(-14+6+12)=\pm 4$$

45 Sol. (D)

Given $P^T=2P+I$.

$$\Rightarrow P=2P^T+I=2(2P+I)+I$$

$$\Rightarrow P+I=0$$

$$\Rightarrow PX+X=0$$

$$PX=-X.$$

46 Sol. (B)

Let $1+a=y$

$$\Rightarrow \left(y^{1/3}-1\right)x^2+\left(y^{1/2}-1\right)x+y^{1/6-1}=0$$

$$\Rightarrow \left(\frac{y^{1/3}-1}{y-1}\right)x^2+\left(\frac{y^{1/2}-1}{y-1}\right)x+\frac{y^{1/6}-1}{y-1}=0$$

Now taking $\lim\limits_{y\to 1}$ on both the sides

$$\Rightarrow \frac{1}{3}x^2+\frac{1}{2}x+\frac{1}{6}=0$$

$$\Rightarrow 2x^2+3x+1=0$$

$$x=-1,-\frac{1}{2}.$$

47 Sol. (A)

$$\text{Required probability}=1-\frac{6.5^3}{6^4}=1-\frac{125}{216}=\frac{91}{216}$$

48 Sol. (B)

$$\int_{-\pi/2}^{\pi/2}\left\{x^2+\ln\left(\frac{\pi+x}{\pi-x}\right)\right\}\cos x dx$$

$$=\int_{-\pi/2}^{\pi/2}x^2\cos x dx+\int_{-\pi/2}^{\pi/2}\ln\left(\frac{\pi+x}{\pi-x}\right)\cos x\, dx$$

$$=2\int_{-\pi/2}^{\pi/2}x^2\cos x\, dx$$

$$=2\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}$$

$$=2\left[\frac{\pi^2}{4}-2\right]=\frac{\pi^2}{2}-4$$

A tangent PT is drawn to the circle $x^2+y^2=4$ at the point $P\left(\sqrt{3},1\right)$. A straight line L, perpendicular to PT is a tangent to the circle $(x-3)^2+y^2=1$.

49 Sol. (A)

Equation of tangent at $P\left(\sqrt{3},1\right)$

$\sqrt{3}x+y=4$

Slope of line perpendicular to the tangent is $\frac{1}{\sqrt{3}}$

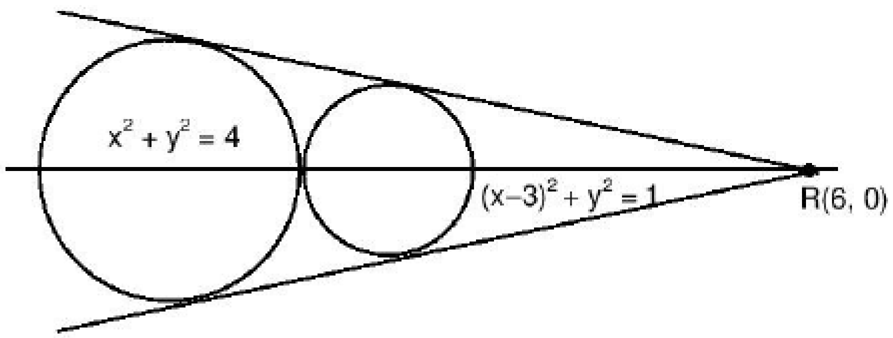
So equation of tangent with slope $\frac{1}{\sqrt{3}}$ to $(x-3)^2+y^2=1$ will be

$$y=\frac{1}{\sqrt{3}}(x-3)\pm1\sqrt{1+\frac{1}{3}}$$
$$\sqrt{3}y=x-3\pm(2)$$

$\sqrt{3}y=x-1$ or $\sqrt{3}y=x-5$

50 Sol. (D)

Point of intersection of direct common tangent is $(6,0)$



so let the equation of common tangent be

$y-0=3(x-6)$

As it touches $x^2+y^2=4$

$$\Rightarrow\left|\frac{0-0+6m}{\sqrt{1+m^2}}\right|=2$$

$9m^2 = 1 + m^2$

$m = \pm\dfrac{1}{2\sqrt{2}}$

So equation of common tangent $y = \dfrac{1}{2\sqrt{2}}(x-6), y = -\dfrac{1}{2\sqrt{2}}(x-6)$ and also $x = 2$

Let $f(x) = (1-x)^2 \sin^2 x + x^2$ for all $x \in IR,$ and let $g(x) = \int_1^x \left(\dfrac{2(t-1)}{t+1} - \ln t\right) f(t)\,dt$ for all $x \in (1, \infty)$

51 Sol. (C)

$$f(x) = (1-x)^2 \sin^2 x + x^2 \quad \forall x \in R$$
$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)\,dt \quad \forall x \in (1, \infty)$$

For statement P:

$$f(x) + 2x = 2(1+x^2) \quad \ldots(\text{i})$$
$$(1-x^2)\sin^2 x + x^2 + 2x = 2 + 2x^2$$
$$(1-x)^2 \sin^2 x = x^2 - 2x + 2 = (x-1)^2 + 1$$
$$(1-x)^2(\sin^2 x - 1) = 1$$
$$-(1-x)^2 \cos^2 x = 1$$
$$(1-x)^2 \cdot \cos^2 x = -1$$

So equitation (i) will not have real solution

So, P is wrong.

For statement Q:

$$2(1-x)^2 \sin^2 x + 2x^2 + 1 = 2x + 2x^2 \quad \ldots(\text{ii})$$
$$2(1-x)^2 \sin^2 x = 2x - 1$$

$2\sin^2 x = \dfrac{2x-1}{(1-x)^2}$ Let $h(x) = \dfrac{2x-1}{(1-x)^2} - 2\sin^2 x$

Clearly $h(0)=-ve, \lim_{x\to1^-} h(x)=+\infty$

So by IVT, equitation (ii) will have solution.

So, Q is correct.

52 Sol. (B)

$g'(x)=\left(\dfrac{2(x-1)}{(x+1)}-\ln x\right)f(x)$. For $x\in(1,\infty)$, $f(x)>0$

Let $h(x)=\left(\dfrac{2(x-1)}{x+1}-\ln x\right)\Rightarrow h'(x)=\left(\dfrac{4}{(x+1)^2}-\dfrac{1}{x}\right)=\dfrac{-(x-1)^2}{(x+1)^2 x}<0$

Also $h(1)=0$ so, $h(x)<0\ \forall x>1$

$\Rightarrow g(x)$ is decreasing on $(1,\infty)$.

Paragraph for Questions 53 and 54

Let a_n denote the number of all n- digit positive integers formed by the digits $0,1$ or both such that no consecutive digits in them are 0.

Let b_n = *the number of such n-digit integers ending with digit* 1 and
c_n = *the number of such n-digit integers ending with digit* 0.

53 Sol. (B)

$a_n=b_n+c_n$
$b_n=a_{n-1}$
$c_n=a_{n-2}\Rightarrow a_n=a_{n-1}+a_{n-2}$

As $a_1=1, a_2=2, a_3=3, a_4=5, a_5=8\Rightarrow b_6=8$

54 Sol. (A)

As $a_n=a_{n-1}+a_{n-2}$

for $n=17$

$\Rightarrow a_{17}=a_{16}+a_{15}$

55 Sol. (B, D)

At $x=2n$

$$L.H.L=\lim_{h\to 0}\left(b_n+\cos\pi\left(2n-h\right)\right)=b_n+1$$
$$R.H.L=\lim_{h\to 0}\left(a_n+\sin\pi\left(2n+h\right)\right)=a_n$$
$$f\left(2n\right)=a_n$$

For continuity $b_n+1=a_n$

At $x=2n+1$

$$L.H.L=\lim_{h\to 0}\left(a_n+\sin\pi\left(2n+1-h\right)\right)=a_n$$
$$R.H.L=\lim_{h\to 0}\left(b_{n+1}+\cos\left(\pi\left(2n+1-h\right)\right)\right)=b_{n+1}-1$$

$$f\left(2n+1\right)=a_n$$

For continuity

$$a_n=b_{n+1}-1$$
$$a_{n-1}-b_n=-1.$$

56 Sol. (B, C)

For given lines to the coplanar, we get

$$\begin{vmatrix}2 & k & 2\\5 & 2 & k\\2 & 0 & 0\end{vmatrix}=0\Rightarrow k^2=4, k=\pm 2$$

For $k=2,$ obviously the plane $y+1=z$ is common in both lines

For $k=-2,$ family of plane containing first lines $x+y+\lambda\left(x-z-1\right)=0$

Point $\left(-1,-1,0\right)$ must satisfy it

$$-2+\lambda\left(-2\right)=0\Rightarrow\lambda=-1$$
$$\Rightarrow y+z+1=0$$

57 Sol. (A, D)

$|Adj\,P| = |P|^2$ **as** $\left(|Adj(P)| = |P|^{n-1}\right)$

Since $|Adj\,P| = 1(3-7) - 4(6-7) + 4(2-1) = 4$

$|P| = 2$ or -2

58 Sol. (A, B)

For $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Let $\cos 4\theta = 1/3$

$$\Rightarrow \cos 2\theta = \pm\sqrt{\frac{1+\cos 4\theta}{2}} = \pm\sqrt{\frac{2}{3}}$$

$$f\left(\frac{1}{3}\right) = \frac{2}{2-\sec^2\theta} = \frac{2\cos^2\theta}{2\cos^2\theta - 1} = 1 + \frac{1}{\cos 2\theta}$$

$$f\left(\frac{1}{3}\right) = 1 - \sqrt{\frac{3}{2}} \text{ or } 1 + \sqrt{\frac{3}{2}}$$

59 Sol. (A,B)

$$P\left(\frac{X}{Y}\right) = \frac{P(X \cap Y)}{P(Y)} = \frac{1}{2} \text{ and } \frac{P(X \cap Y)}{P(X)} = \frac{1}{3}$$

$$P(X \cap Y) = \frac{1}{6} \Rightarrow P(Y) = \frac{1}{3} \text{ and } P(X) = \frac{1}{2}$$

Clearly, X and Y are independent

Also, $P(X \cup Y) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$

60 Sol. (A, B,C, D)

$f'(x) = e^{x^2}(x-2)(x-3)$

Clearly, maximum at $x=2$, minima at $x=3$ and

decreasing in $x \in (2,3)$.

$f'(x)=0$ for $x=2$ and $x=3$ (Rolle's theorem)

So there exist $c \in (2,3)$ for which

$f''(c)=0.$