JEE MAIN – 2020

MATHEMATICS

SECTION A

51: Sol. (2)

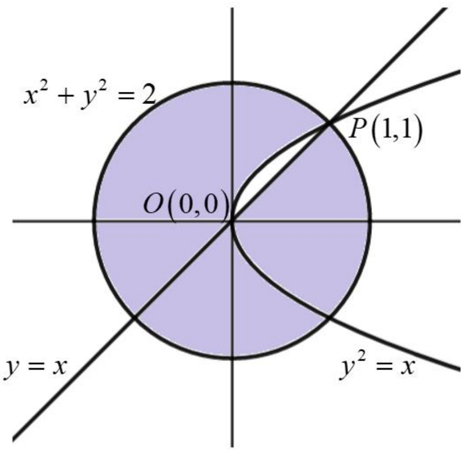
Required area $=$ Area of the circle $-$ Area bounded by given line and parabola

Required area $=\pi r^2-\int_0^1\left(y-y^2\right)dy$

$$=2\pi-\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1$$

$$=2\pi-\frac{1}{6}$$

$$=\frac{1}{6}(12\pi-1)$$



52: : Sol. (4)

Selecting all 5 digits $={}^5C_5=1$ way

Now, we need to select one more digit to make it a 6 digit number $={}^5C_1=5$ ways

Total number of permutations $=\frac{6!}{2!}$

Total numbers $={}^5C_5\times{}^5C_1\times\frac{6!}{2!}=\frac{5}{2}(6!)$

53: Sol. (1)

$k =$ no. of consecutive heads

When $k = 3$, we have $\{\text{HHHTH, HHHTT, THHHT, HTHHH, TTHHH}\}$

$\therefore P(3) = \frac{5}{32}$

When $k = 4$, we have $\{\text{HHHHT, THHHH}\}$

$\therefore P(4) = \frac{2}{32}$

When $k = 5$, we have $\{\text{HHHHH}\}$

$\therefore P(5) = \frac{1}{32}$

$$P(\bar{3} \cap \bar{4} \cap \bar{5}) = 1 - \left(\frac{5}{32} + \frac{2}{32} + \frac{1}{32}\right) = \frac{24}{32}$$

$$\sum XP(X) = \left(-1 \times \frac{24}{32}\right) + \left(3 \times \frac{5}{32}\right) + \left(4 \times \frac{2}{32}\right) + \left(5 \times \frac{1}{32}\right)$$

$$= \frac{1}{8}$$

54: Sol. (3)

$z = x + iy$

$$\frac{z-1}{2z+i} = \frac{x+iy-1}{2x+2iy+i}$$

$$= \frac{(x-1)+iy}{2x+i(2y+1)} \times \frac{2x-i(2y+1)}{2x-i(2y+1)}$$

$$= \frac{2x(x-1)+y(2y+1)}{4x^2+(2y+1)^2}$$

$$\therefore \frac{2x(x-1)+y(2y+1)}{4x^2+(2y+1)^2} = 1$$

$$\Rightarrow 2x^2 - 2x + 2y^2 + y = 4x^2 + 4y^2 + 4y + 1$$

$$\Rightarrow -2x^2 - 2y^2 - 2x - 3y - 1 = 0$$

$$\Rightarrow x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0$$

Circle's centre will be $\left(-\frac{1}{2},-\frac{3}{4}\right)$

Radius $=\sqrt{\frac{1}{4}+\frac{9}{16}-\frac{1}{2}}=\sqrt{\frac{4+9-8}{16}}=\frac{\sqrt{5}}{4}$

Diameter $=\frac{\sqrt{5}}{2}$

55: Sol. (3)

$$f(a+b+1-x)=f(x) \qquad \ldots(1)$$
$$f(a+b-x)=f(x+1) \qquad \ldots(2)$$

$$I=\frac{1}{(a+b)}\int_a^b x\left[f(x)+f(x+1)\right]dx \qquad \ldots(3)$$

From (1) and (2)

$$I=\frac{1}{(a+b)}\int_a^b (a+b-x)\left[f(x+1)+f(x)\right]dx \qquad \ldots(4)$$

Adding (3) and (4)

$$2I=\int_a^b\left[f(x)+f(x+1)\right]dx$$

$$2I=\int_a^b f(a+b+1-x)dx+\int_a^b f(x)dx$$

$$2I=2\int_a^b f(x)dx$$

$$I=\int_a^b f(x)dx$$

$$I=\int_{a-1}^{b-1} f(t+1)dt \qquad ;\ x=t+1,\ dx=dt$$

$$I=\int_{a-1}^{b-1} f(x+1)dx$$

56: Sol. (4)

Let the equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$a>b$$

Now $2ae = 6$ and $\frac{2a}{e} = 12$

$\therefore ae = 3$ and $\frac{a}{e} = 6$

$\Rightarrow a^2 = 18$

$\therefore a^2e^2 = c^2 = a^2 - b^2 = 9$

$\Rightarrow b^2 = 9$

length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{\sqrt{18}} = 3\sqrt{2}$

57: Sol. (1)

p	q	$p \Rightarrow q$	$\sim p$	$q \Rightarrow \sim p$	$(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$
T	T	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	T	T

Clearly $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to $\sim p$.

58: Sol. (4)

$1 + 49 + 49^2 + \ldots + 49^{125}$

$= \frac{49^{126} - 1}{49 - 1}$

$= \frac{(49^{63} + 1)(49^{63} - 1)}{48}$

$= \frac{(49^{63} + 1)\left[(1 + 48)^{63} - 1\right]}{48}$

$= \frac{(49^{63} + 1)[1 + 48I - 1]}{48}$; where I is an integer

$=\left(49^{63}+1\right)I$

$\therefore$ Greatest positive integer is $k=63$

59: Sol. (4)

Angle bisector can be $\vec{a}=\lambda\left(\vec{b}+\vec{c}\right)$ or $\vec{a}=\mu\left(\vec{b}-\vec{c}\right)$

$$\vec{a}=\lambda\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}+\frac{\hat{i}-\hat{j}+4\hat{k}}{3\sqrt{2}}\right)$$

$$=\frac{\lambda}{3\sqrt{2}}\left(3\hat{i}+3\hat{j}+\hat{i}-\hat{j}+4\hat{k}\right)$$

$$=\frac{\lambda}{3\sqrt{2}}\left(4\hat{i}+2\hat{j}+4\hat{k}\right) \qquad \ldots(1)$$

Comparing this with $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$ we get

$$\frac{2\lambda}{3\sqrt{2}}=2\Rightarrow\lambda=3\sqrt{2}$$

Putting $\lambda=3\sqrt{2}$ in equation (1) we get,

$$\vec{a}=4\hat{i}+2\hat{j}+4\hat{k}$$

Now,

$$\vec{a}=\mu\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}-\frac{\hat{i}-\hat{j}+4\hat{k}}{3\sqrt{2}}\right)$$

$$=\frac{\mu}{3\sqrt{2}}\left(3\hat{i}+3\hat{j}-\hat{i}+\hat{j}-4\hat{k}\right)$$

$$=\frac{\mu}{3\sqrt{2}}\left(2\hat{i}+4\hat{j}-4\hat{k}\right)$$

$$=\frac{2\mu}{3\sqrt{2}}\left(\hat{i}+2\hat{j}-2\hat{k}\right) \qquad \ldots(2)$$

Comparing this with $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$ we get

$$\frac{2\mu}{3\sqrt{2}}=1\Rightarrow\mu=\frac{3\sqrt{2}}{2}$$

Putting $\mu=\frac{3\sqrt{2}}{2}$ in equation (2) we get,

$$\vec{a}=\hat{i}+2\hat{j}-2\hat{k}$$

$\therefore \vec{a}\cdot k = -2 \quad \Rightarrow \vec{a}\cdot k + 2 = 0$

60: Sol. (3)

$$y(\alpha) = \sqrt{2\left(\frac{\tan\alpha+\cot\alpha}{1+\tan^2\alpha}\right)+\frac{1}{\sin^2\alpha}}$$

$$= \sqrt{2\left(\frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}}{1+\frac{\sin^2\alpha}{\cos^2\alpha}}\right)+\frac{1}{\sin^2\alpha}}$$

$$= \sqrt{2\left(\frac{1}{\sin\alpha\cos\alpha}\times\frac{\cos^2\alpha}{1}\right)+\frac{1}{\sin^2\alpha}}$$

$$= \sqrt{2\cot\alpha+\operatorname{cosec}^2\alpha}$$

$$= \sqrt{(1+\cot\alpha)^2}$$

$$= -1-\cot\alpha$$

$$\therefore \frac{dy}{d\alpha} = 0+\operatorname{cosec}^2\alpha$$

$$= \operatorname{cosec}^2\frac{5\pi}{6}$$

$$= 4$$

61: Sol. (3)

Any tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{a}{m}$

Comparing it with $y = mx + 4$, we get $\frac{1}{m} = 4$

So $m = \frac{1}{4}$.

Equation of tangent becomes $y = \frac{1}{4}x + 4$

$y = \frac{1}{4}x + 4$ is a tangent to $x^2 = 2by$

$x^2 = 2b\left(\frac{1}{4}x + 4\right)$

$\Rightarrow 4x^2 = 2bx + 32b$

$\Rightarrow 2x^2 - bx - 16b = 0$

$D = 0$

$b^2 + 128b = 0$

$b = 0$ (not possible)

$\Rightarrow b = -128$

62: Sol. (3)

The roots of equation $x^2 + x + 1 = 0$ are complex cube roots of unity $= \omega$ or ω^2

$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix}$$

$$A^2 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix}$$

$$A^2 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = I$$

$A^{28} = I$

Therefore, we get

$A^{31} = A^{28} A^{3}$
$A^{31} = IA^{3}$
$A^{31} = A^{3}$

63: Sol. (2)

$g(x) = x^2 + x - 1$

$(g \circ f)(x) = 4x^2 - 10x + 5$
$\Rightarrow g(f(x)) = 4x^2 - 10x + 5$
$\Rightarrow f^2(x) + f(x) - 1 = 4x^2 - 10x + 5$

Putting $x = \frac{5}{4}$ and $f\left(\frac{5}{4}\right) = t$ we get

$t^2 + t - 1 = 4\left(\frac{5}{4}\right)^2 - 10 \times \frac{5}{4} + 5$
$\Rightarrow t^2 + t - 1 = \frac{25 - 50 + 20}{4}$
$\Rightarrow t^2 + t + \frac{1}{4} = 0$
$\Rightarrow t = -\frac{1}{2}$
$\therefore f\left(\frac{5}{4}\right) = -\frac{1}{2}$

64: Sol. (3)

$(k+1)\tan^2 x - \sqrt{2}\lambda \tan x = 1 - k$

$\tan^2(\alpha + \beta) = 50 \qquad \ldots(1)$

$\because \tan\alpha$ and $\tan\beta$ are the roots of the given equation

Now

$\tan\alpha + \tan\beta = \frac{\sqrt{2}\lambda}{(k+1)}$,

$\tan\alpha \tan\beta = \frac{(k-1)}{(k+1)}$

$$\therefore \left(\frac{\frac{\sqrt{2}\lambda}{k+1}}{1-\frac{k-1}{k+1}} \right)^2 = 50 \qquad \ldots[\text{from } (1)]$$

$$\Rightarrow \frac{2\lambda^2}{4} = 50$$

$$\Rightarrow \lambda^2 = 100$$

$$\Rightarrow \lambda = 10 \quad \text{or} \quad \lambda = -10$$

65: Sol. (2)

Given

$A(2,\ 1,\ 0)$

$B(4,\ 1,\ 1)$

$C(5,\ 0,\ 1)$

Now

$$\overrightarrow{AB} = (2, 0, 1)$$

$$\overrightarrow{AC} = (3, -1, 1)$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$\Rightarrow \begin{vmatrix} x-2 & y-1 & z \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x-2+y-1-2z = 0$$

$\therefore$ Equation of the plane is $x+y-2z=3$.

Let the image of point $(2,\ 1,\ 6)$ is $(l,\ m,\ n)$.

$$\frac{l-2}{1} = \frac{m-1}{1} = \frac{n-6}{-2} = \frac{-2(2+1-2\times 6-3)}{6} = 4$$

$$\Rightarrow l = 6,\ m = 5,\ n = -2$$

Hence the image of R in the plane P is $(6,\ 5,\ -2)$.

66: Sol. (3)

$x^k + y^k = a^k$

$kx^{k-1} + ky^{k-1}\left(\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{k-1}$

$\frac{dy}{dx} + \left(\frac{y}{x}\right)^{1-k} = 0$

$1 - k = \frac{1}{3}$

$k = \frac{2}{3}$

67: Sol. (2)

$f(-7) = -3$ and $f'(x) \le 2$

Applying LMVT in $[-7,\ 0]$, we get

$\frac{f(-7) - f(0)}{-7} = f'(c) \le 2$

$\Rightarrow \frac{-3 - f(0)}{-7} \le 2$

$\Rightarrow f(0) + 3 \le 14$

$\Rightarrow f(0) \le 11$

Applying LMVT in $[-7,\ -1]$, we get

$\frac{f(-7) - f(-1)}{-6} = f'(c) \le 2$

$\Rightarrow \frac{-3 - f(-1)}{-6} \le 2$

$\Rightarrow f(-1) + 3 \le 12$

$\Rightarrow f(-1) \le 9$

Therefore $f(-1) + f(0) \le 20$.

68: Sol. (4)

$$e^y\left(\frac{dy}{dx}-1\right)=e^x$$

$$\Rightarrow \frac{dy}{dx}=e^{x-y}+1$$

Let $x-y=t$

$$\Rightarrow 1-\frac{dy}{dx}=\frac{dt}{dx}$$

So, we can write

$$1-\frac{dt}{dx}=e^t+1$$

$$\Rightarrow -e^{-t}dt=dx$$

$$\Rightarrow -e^{-t}=x+c$$

$$\Rightarrow -e^{y-x}=x+c$$

$$1=0+c$$

$$\Rightarrow -e^{y-x}=x+1$$

at $x=1$

$$\Rightarrow e^{y-1}=\ 2$$

$$\Rightarrow y=1+\log_e 2$$

69: Sol. (1)

Let 5 numbers be $a-2d,\ a-d,\ a,\ a+d,\ a+2d$

$$5a=25$$

$$\Rightarrow a=5$$

$$(a-2d)(a-d)(a)(a+d)(a+2d)=2520$$

$$\Rightarrow (25-4d^2)(25-d^2)=504$$

$$\Rightarrow 4d^4-125d^2+121=0$$

$$\Rightarrow 4d^4-4d^2-121d^2+121=0 \ \ (25-4d^2)(25-d^2)=504$$

$$\Rightarrow d^2=1 \quad \text{or} \quad d^2=\frac{121}{4}$$

$$\Rightarrow d=\frac{11}{2}$$

For $d=\frac{11}{2}$, $a+2d$ is the greatest term, $a+2d=5+11=16$.

70: Sol. (3)

$$\begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{vmatrix} 2 & 2a & a \\ 0 & 3b-2a & b-a \\ 0 & 4c-2a & c-a \end{vmatrix} = 0$$

$$\Rightarrow (3b-2a)(c-a)-(4c-2a)(b-a)=0$$

$$\Rightarrow 3bc-2ac-3ab-2a^2-\left(4bc-2ab-4ac+2a^2\right)=0$$

$$\Rightarrow -bc+2ac-ab=0$$

$$\Rightarrow ab+bc=2ac$$

$$\Rightarrow b(a+c)=2ac$$

$$\Rightarrow b=\frac{2}{\frac{1}{a}+\frac{1}{c}}$$

$$\Rightarrow \frac{1}{a}+\frac{1}{c}=\frac{2}{b}$$

Hence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

SECTION B

71: Sol. 36

$$\lim_{x\to 2}\frac{3^x+3^{3-x}-12}{3^{-\frac{x}{2}}-3^{1-x}} = \lim_{x\to 2}\frac{3^x+\frac{27}{3^x}-12}{\frac{1}{3^{\frac{x}{2}}}-\frac{3}{3^x}}$$

$$=\lim_{t\to 3}\frac{t^2+\frac{27}{t^2}-12}{\frac{1}{t}-\frac{3}{t^2}} \qquad \ldots\left(\text{put } 3^{\frac{x}{2}}=t\right)$$

$$=\lim_{t\to 3}\frac{\left(t^2-9\right)\left(t^2-3\right)}{(t-3)}$$

$$=\lim_{t\to 3}\left(t^2-3\right)(t+3)$$

$$=36$$

72: Sol. 18

For n natural number variance is given by

$$\sigma^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

$$\frac{\sum x_i^2}{n} = \frac{1^2 + 2^2 + \ldots + n^2}{n} = \frac{n(n+1)(2n+1)}{6n}$$

$$\frac{\sum x_i}{n} = \frac{1+2+\ldots+n}{n} = \frac{n(n+1)}{2n}$$

$$\therefore \sigma^2 = \frac{n^2-1}{12} = 10$$

$$\Rightarrow n = 11$$

Variance of $(2,\ 4,\ 6\ldots) = 4\times$ variance of $(1,2,3,4\ldots)$

$$= 4\times\frac{m^2-1}{12}$$

$$= \frac{m^2-1}{3}$$

But variance of first m even natural numbers is 16.

$$\therefore \frac{m^2-1}{3} = 16$$

$$\Rightarrow m = 7$$

Therefore, $n+m = 11+7 = 18$.

73: Sol. 30

Let $(1+x+x^2+x^3+\ldots+x^{2n})(1-x+x^2-x^3+\ldots+x^{2n}) = a_0 + a_1x + a_2x^2 + \ldots$

Put $x=1$

$$2n+1 = a_0 + a_1 + a_2 + a_3 + \ldots \qquad \ldots(1)$$

Put $x=-1$

$$2n+1 = a_0 - a_1 + a_2 - a_3 + \ldots \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2(2n+1) = 2(a_0 + a_2 + a_4 + \ldots)$$

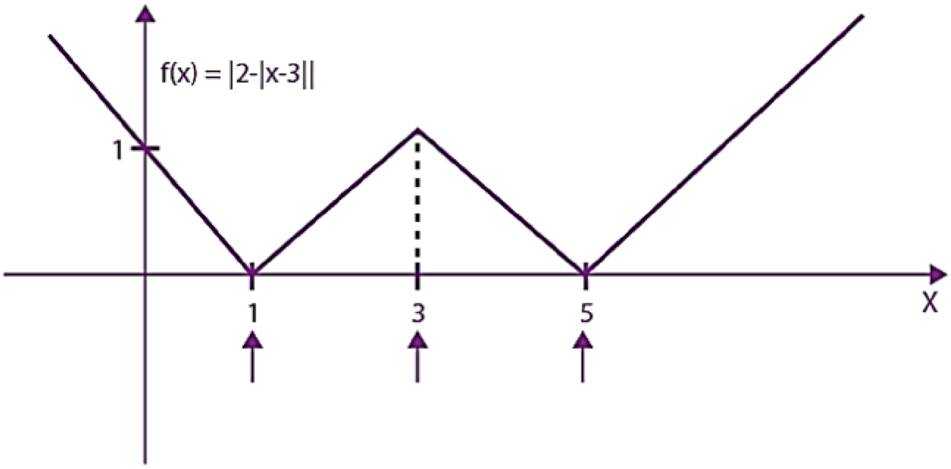
$$\Rightarrow 2n+1 = 61$$

$$\Rightarrow n = 30$$

74: Sol. 3

There will be three points $x = 1, 3, 5$ at which $f(x)$ is non-differentiable.

$$\therefore f(f(1)) + f(f(3)) + f(f(5))$$
$$= f(0) + f(2) + f(0)$$
$$= 1 + 1 + 1$$
$$= 3$$



75: Sol. 5

Correct Answer: 5

P is the centroid which is $= \left(\dfrac{1 + 6 + \frac{3}{2}}{3}, \dfrac{0 + 2 + 6}{3} \right)$

$$P = \left(\frac{17}{6}, \frac{8}{3} \right)$$

$$Q = \left(-\frac{7}{6}, -\frac{1}{3} \right)$$

$$PQ = \sqrt{4^2 + 3^2} = 5$$