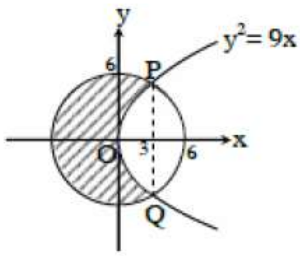
JEE MAIN-2021

MATHEMATICS

31: Sol. (4)

The curves intersect at points $\left(3, \pm 3\sqrt{3}\right)$



Required area

$$= \pi r^2 - 2\left[\int_0^3 \sqrt{9x}\,dx + \int_3^6 \sqrt{36 - x^2}\,dx\right]$$

$$= 36\pi - 12\sqrt{3} - 2\left|\frac{x}{2}\sqrt{36 - x^2} + 18\sin^{-1}\left(\frac{x}{6}\right)\right|_3^6$$

$$= 36\pi - 12\sqrt{3} - 2\left\{9\pi - \left(\frac{9\sqrt{3}}{2} + 3\pi\right)\right\}$$

$$= 24\pi - 3\sqrt{3}$$

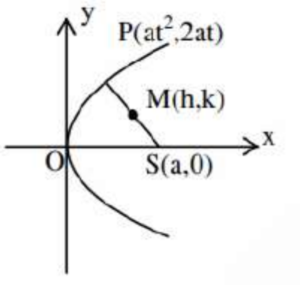
32. Sol. (2)

The locus of the midpoint of the line segment joining the focus of the moving point on the parabola $y^2 = 4ax$ is another parabola with directrix:

Let midpoint be $M\left(h, k\right)$, Let moving point be $\left(at^2, 2at^2\right)$ and let focus $S\left(a, 0\right)$

$$\therefore h = \frac{at^2 + a}{2}, k = \frac{2at + 0}{2}$$

$$\therefore t^2 = \frac{2h - a}{a} \text{ and } t = \frac{k}{a}$$



$\Rightarrow \frac{k^2}{a^2} = \frac{2h-a}{a}$

$\Rightarrow$ Locus of (h,k) is $y^2 = a(2x-a)$

$\Rightarrow y^2 = 2a\left(x - \frac{a}{2}\right)$

Its directrix is $x - \frac{a}{2} = -\frac{a}{2} \Rightarrow x = 0$

33. Sol: (1)

Normal vector of required plane is $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 2 & -5 & -1 \end{vmatrix} = -11\hat{i} - \hat{j} - 17\hat{k}$

Therefore, equation of plane is:

$11(x-1)+(y-2)+17(z+3)=0$

$\therefore 11x + y + 17z + 38 = 0$

34. Sol. (1)

Given that there are 6 Indians 8 foreigners

We have to find number of committee form with at least 2 Indians such numbers of foreigners is twice the number of Indians.

So, the possible cases are $(2I, 4F), (3I, 6F), (4I, 8F)$.

Therefore, number of ways is:

$(2I,4F)+(3I,6F)+(4I,8F)$

$= {}^6C_2\,{}^8C_4 + {}^6C_3\,{}^8C_6 + {}^6C_4\,{}^8C_8$

$= 15\times 70 + 20\times 28 + 15\times 1$

$= 1050 + 560 + 15$

$= 1625$

35. Sol. (1)

We have function $f(x) = [x-1]\cos\left(\frac{2x-1}{2}\right)\pi$

Doubtful points are, $x = n, n \in I$

$$\text{L.H.L} = \lim_{x\to n^-}[x-1]\cos\left(\frac{2x-1}{2}\right)\pi = (n-2)\cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$$\text{R.H.L} = \lim_{x\to n^+}[x-1]\cos\left(\frac{2x-1}{2}\right)\pi = (n-1)\cos\left(\frac{2n-1}{2}\right)\pi = 0$$

And $f(n) = 0$

Hence continuous for every real x .

36: Sol. (4)

we have, $p+q=2$ and $p^4+q^4=272$.

$\therefore (p^2+q^2)^2 - 2p^2q^2 = 272$

$\therefore \left\{(p+q)^2 - 2pq\right\}^2 - 2p^2q^2 = 272$

$\therefore \left\{(p+q)^4 + 4p^2q^2 - 2(p+q)^2(2pq)\right\} - 2p^2q^2 = 272$

$\therefore 16 + 4p^2q^2 - 16pq - 2p^2q^2 = 272$

$\therefore 16 - 16pq + 2p^2q^2 = 272$

$\therefore (pq)^2 - 8pq - 128 = 0$

$\therefore\ pq = \frac{8 \pm 24}{2} = 16, -8$

Since, roots are positive

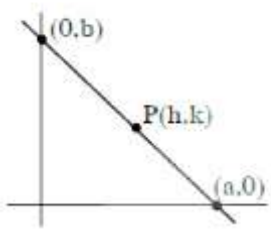
$\therefore\ pq = 16$

Now, the quadratic equation whose roots are p and q is:

$x^2 - (p + q)x + pq = 0$

$\Rightarrow\ x^2 - 2x + 16 = 0$

37. Sol. (1)



Equation of line in intercept form is $\frac{x}{a} + \frac{y}{b} = 1$

Let the point $P(h,k)$ is moving on the straight line as shown in figure.

$\therefore\ \frac{h}{a} + \frac{k}{b} = 1 \qquad \cdots(\text{i})$

It is given that $\frac{\frac{1}{a} + \frac{1}{b}}{2} = \frac{1}{4}$

$\therefore\ \frac{1}{a} + \frac{1}{b} = \frac{1}{2} \qquad \cdots(\text{ii})$

$\therefore$ Line passes through fixed point $B(2,2)$.[from (i) and (ii)]

38. Sol. (3)

$\because\ P$ (odd no. twice) $= P$ (even no. thrice)

$$\Rightarrow {}^nC_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{n-2} = {}^nC_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{n-3}$$

$$\Rightarrow {}^nC_2\left(\frac{1}{2}\right)^n = {}^nC_3\left(\frac{1}{2}\right)^n$$

$$\Rightarrow n = 5$$

Success is getting an odd number then P (Odd successes) $= P(1) + P(3) + P(5)$

$$= {}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5$$

$$= \frac{16}{2^5}$$

$$= \frac{1}{2}$$

39. Sol. (1)

$$e^{\left(\cos^2\theta+\cos^4\theta+\cdots\infty\right)\ln 2} = 2^{\cos^2\theta+\cos^4\theta+\cdots\infty}$$

$$= 2^{\left(\frac{\cos^2\theta}{1-\cos^2\theta}\right)}$$

$$= 2^{\cot^2\theta}$$

Now,

$$t^2 - 9t + 8 = 0$$

$$\Rightarrow (t-1)(t-8) = 0$$

$$\Rightarrow t = 1, 8$$

Since, $2^{\cot^2\theta}$ is a root of $t^2 - 9t + 8$

$$\therefore 2\cot^2\theta = 1, 8$$

$$\Rightarrow \cot^2\theta = 0, 3$$

But, $0 < \theta < \frac{\pi}{2} \Rightarrow \cot\theta = \sqrt{3}$

$$\Rightarrow \frac{2\sin\theta}{\sin\theta + \sqrt{3}\cos\theta} = \frac{2}{1+\sqrt{3}\cot\theta} = \frac{2}{4} = \frac{1}{2}$$

40. Sol. (3)

$$\frac{dP(t)}{dt} = (0.5)P(t) - 450$$

$$\frac{dP(t)}{dt} = \frac{P(t) - 900}{2}$$

$$\int_0^t \frac{dP(t)}{P(t) - 900} = \int_0^t \frac{dt}{2}$$

$$\left\{\ln\left|P(t) - 900\right|\right\}_0^t = \left\{\frac{t}{2}\right\}_0^t$$

$$\ln\left|P(t) - 900\right| - \ln\left|P(0) - 900\right| = \frac{t}{2}$$

$$\ln\left|P(t) - 900\right| - \ln 50 = \frac{t}{2} \qquad \left[\because P(0) = 850\right]$$

Let at $t = t_1, P(t) = 0$ hence

$$\ln\left|P(t) - 900\right| - \ln 50 = \frac{t_1}{2}$$

$$\ln 900 - \ln 50 = \frac{t_1}{2}$$

$$\ln\left(\frac{900}{50}\right) = \frac{t_1}{2}$$

$$t_1 = 2\ln 18$$

41. Sol. (1)

$$I = \int \frac{\cos\theta - \sin\theta}{\sqrt{8 - \sin 2\theta}} d\theta = a\sin^{-1}\left(\frac{\sin\theta + \cos\theta}{b}\right) + C \qquad ....(i)$$

Put $\sin\theta + \cos\theta = t \Rightarrow 1 + \sin 2\theta = t^2$

$$(\cos\theta - \sin\theta)\,d\theta = dt$$

$$\therefore I=\int \frac{dt}{\sqrt{8-\left(t^2-1\right)}}$$

$$=\int \frac{dt}{\sqrt{9-t^2}}$$

$$=\sin^{-1}\left(\frac{t}{3}\right)+C$$

$$=\sin^{-1}\left(\frac{\sin\theta+\cos\theta}{3}\right)+C$$

From equation (i), we get $a=1$ and $b=3$.

42. Sol. (1)

$$A\wedge(\sim A\vee B)\rightarrow B$$

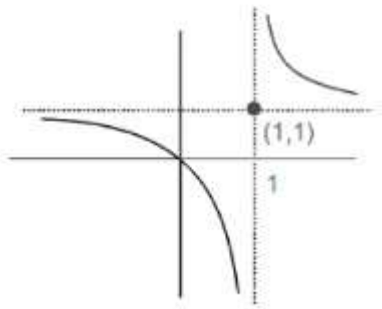
$$=\left[(A\wedge\sim A)\vee(A\wedge B)\right]\rightarrow B$$

$$=(A\wedge B)\rightarrow B$$

$$=\sim A\vee\sim B\vee B$$

$$=t$$

43. Sol. (3)



We have, $f(x)=2x-1, g(x)=\frac{x-\frac{1}{2}}{x-1}$

$$f(g(x)) = 2g(x) - 1$$

$$= 2\frac{\left(x - \frac{1}{2}\right)}{x-1}$$

$$= \frac{x}{x-1}$$

$$= 1 + \frac{1}{x-1}$$

Hence, $f(g(x))$ is one-one, into

44. Sol. (2)

$$S_1 = -{}^{15}C_1 + 2.{}^{15}C_2 - \cdots - 15.{}^{15}C_{15}$$

$$= \sum_{r=1}^{15} (-1)^r \cdot {}^{15}C_r$$

$$= 15\sum_{r}^{15} (-1)^r \cdot {}^{14}C_{r-1}$$

$$= 15\left(-{}^{14}C_0 + {}^{14}C_1 - \cdots - {}^{14}C_{14}\right)$$

$$= 15(0)$$

$$= 0$$

$$S_2 = {}^{14}C_1 + {}^{14}C_3 + \cdots + {}^{14}C_{11}$$

$$= \left({}^{14}C_1 + {}^{14}C_3 + \cdots + {}^{14}C_{11} + {}^{14}C_{13}\right) - {}^{14}C_{13}$$

$$= 2^{13} - 14$$

$$\therefore S_1 + S_2 = 2^{13} - 14$$

45.Sol. (1)

$$\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2} = \lambda$$

$$\Rightarrow x = \lambda + 3, y = 2\lambda + 4, z = 2\lambda + 5$$

Which lies on given plane hence

$\Rightarrow \lambda + 3 + 2\lambda + 4 + 2\lambda + 5 = 17$

$\Rightarrow \lambda = \frac{5}{5} = 1$

Hence, point of intersection is $Q(4, 6, 7)$

$\therefore$ Required distance $= PQ$

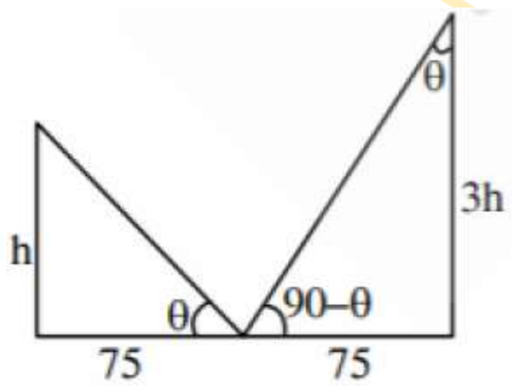
$= \sqrt{(4-1)^2 + (6-1)^2 + (7-9)^2}$

$= \sqrt{3^2 + 5^2 + (-2)^2}$

$= \sqrt{9 + 25 + 4}$

$= \sqrt{38}$

46: Sol. (1)



$\tan\theta = \frac{h}{75} = \frac{75}{3h}$

$\Rightarrow h^2 = \frac{(75)^2}{3}$

$\Rightarrow h = 25\sqrt{3}$ m

47. Sol. (2)

$$\lim_{x\to 0^+}\frac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x^3} = \lim_{x\to 0^+}\frac{(\sin|x|)\,2x}{3x^2}$$ [using leibniz's theorem]

$$= \lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)\times\frac{2}{3}$$

$$= \frac{2}{3}$$

48. Sol. (3)

Equation of tangent at $P(t, t^3)$

$$(y - t^3) = 3t^2(x - t)\cdots\cdots(\text{i})$$

now solve the above equation with

$$y = x^3\cdots\cdots(\text{ii})$$

By (i) & (ii)

$$x^3 - t^3 = 3t^2(x - t)$$

$$\Rightarrow x^2 + xt + t^2 = 3t^2$$

$$\Rightarrow x^2 + xt - 2t^2 = 0$$

$$\Rightarrow (x - t)(x + 2t) = 0$$

$$\Rightarrow x = t, -2t$$

$$\Rightarrow Q(-2t, 8t^3)$$

$\therefore$ Ordinate of required point $= \dfrac{2t^3 + (-8t^3)}{3} = -2t^3$

49. Sol. (4)

$$\Delta = \begin{vmatrix} 3 & -2 & -k \\ 2 & -4 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 0$$

$\Rightarrow 3(4+4) + 2(-2+2) - k(4+4) = 0$

$\Rightarrow k = 3$

$$\Delta_x = \begin{vmatrix} 10 & -2 & -3 \\ 6 & -4 & -2 \\ 5\text{m} & 2 & -1 \end{vmatrix}$$

$= 10(4+4) + 2(-6+10m) - 3(12+20m)$

$= 80 - 12 + 20m - 36 - 60m$

$= 8(4-5m)$

$$\Delta_y = \begin{vmatrix} 3 & 10 & -3 \\ 2 & 6 & -2 \\ 1 & 5\text{m} & -1 \end{vmatrix}$$

$= 3(-6+10m) - 10(-2+2) - 3(10m-6)$

$= 0$

$$\Delta_z = \begin{vmatrix} 3 & -2 & 10 \\ 2 & -4 & 6 \\ 1 & 2 & 5\text{m} \end{vmatrix}$$

$= 3(-20m-12) + 2(10m-6) + 10(4+4)$

$= -60m - 36 + 20m - 12 + 80$

$= 8(4-5m)$

For inconsistent system $\Delta = 0$ and at least any one of $\Delta_x, \Delta_y, \Delta_z \neq 0$.

$\therefore k = 3$ and $m \neq \frac{4}{5}$

50. Sol. (2)

$$f(x) = \frac{4x^3 - 3x^2}{6} - 2\sin x + (2x-1)\cos x$$

$\therefore\ f'(x)=(2x-1)(x-\sin x)$

$\Rightarrow\ f'(x)\geq 0$ in $x\in\left[\frac{1}{2},\infty\right)$ and $f'(x)\leq 0$ in $x\in\left(-\infty,\frac{1}{2}\right]$

SECTION B

51. Sol. 9

Let $f(x)=\dfrac{4}{\sin x}+\dfrac{1}{1-\sin x}$

$$y=\frac{4-3\sin x}{\sin x(1-\sin x)}$$

Let $\sin x=t\ \because\ x\in\left(0,\frac{\pi}{2}\right)\ \Rightarrow 0<t<1$

$$y=\frac{4-3t}{t-t^2}$$

$$\frac{dy}{dt}=\frac{-3\left(t-t^2\right)-(1-2t)(4-3t)}{\left(t-t^2\right)^2}=0$$

$\Rightarrow 3t^2-3t-\left(4-11t+6t^2\right)=0$

$\Rightarrow 3t^2-8t+4=0$

$\Rightarrow 3t^2-6t-2t+4=0$

$\Rightarrow 3t(t-2)-2(t-2)=0$

$\Rightarrow (t-2)(3t-2)=0$

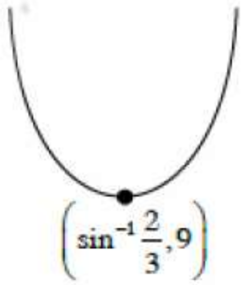
$\Rightarrow t=2,\dfrac{2}{3}$

But $\sin x=2$ is not possible.

$\therefore t=\dfrac{2}{3}$

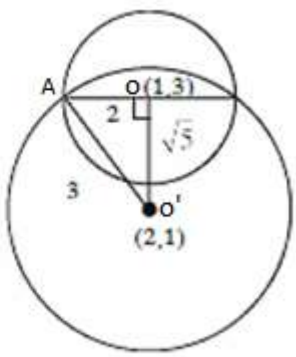
$\Rightarrow \alpha\geq 9$

Least value of α is equal to 9.



52: Sol. 3

Solution:



diameter of circle $C_1 : x^2 + y^2 - 2x - 6y + 6 = 0$

therefore, centre of $C_1 = (1,3)$

let r_1 be the diameter of circle C_1

$\therefore r_1 = \sqrt{g^2 + f^2 - c}$

$\therefore r_1 = \sqrt{(-1)^2 + (-3)^2 - 6}$

$\therefore r_1 = 2$

Centre of C_2 is $(2,1)$.

Distance between $(1,3)$ and $(2,1)$ is $\sqrt{5}$

Let r_2 be the radius of circle C_2.

In $\Delta OO'A$

$$\therefore \left(\sqrt{5}\right)^2 + (2)^2 = r_2^{\,2}$$

$$\Rightarrow r_2 = 3$$

53: Correct Answer: 1

$$\tan\left(\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{1+r^2+r}\right)\right)$$

$$= \tan\left\{\lim_{n\to\infty}\sum_{r=1}^{n}\left[\tan^{-1}(r+1)-\tan^{-1}(r)\right]\right\}$$

$$= \tan\left\{\lim_{n\to\infty}\left(\tan^{-1}(n+1)-\frac{\pi}{4}\right)\right\}$$

$$= \tan\left(\frac{\pi}{4}\right)$$

$$= 1$$

54: Sol. 17

As $PQ = kI \Rightarrow Q = kP^{-1}I$

Now $Q = \frac{k}{|P|}(adj\,P)I = Q = \frac{k}{(20+12\alpha)}\begin{bmatrix} 5\alpha & -10 & -\alpha \\ 3\alpha & -6 & (-3\alpha-4) \\ -10 & 12 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\because q_{23} = \frac{-k}{8}$

$$\Rightarrow \frac{k}{(20+12\alpha)}(-3\alpha-4) = \frac{-k}{8}$$

$$\Rightarrow 2(3\alpha)+4 = 5+3\alpha$$

$$\Rightarrow 3\alpha = -3$$

$$\Rightarrow \alpha = -1$$

Also $|Q| = \frac{k^3|I|}{|P|}$

$\Rightarrow \frac{k^2}{2} = \frac{k^3}{(20+12\alpha)}$

$\Rightarrow (20+12\alpha) = 2k$

$\Rightarrow 10 + 6(-1) = k$

$\Rightarrow k = 4$

$\therefore k^2 + \alpha^2 = 4^2 + (-1)^2$

$= 16 + 1$

$= 17$

55: Sol. 6

Let x, y, z be probability of B_1, B_2, B_3 respectively

The probability that only B_1 occurs

$x(1-y)(1-z) = \alpha$

The probability that only B_2 occurs:

$y(1-x)(1-z) = \beta$

The probability that only B_3 occurs:

$z(1-x)(1-y) = \gamma$

The probability p that none of events B_1, B_2 or B_3 occurs:

$(1-x)(1-y)(1-z) = p$

We have relations $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$

Putting the value of α, β and γ in the given relation,

we get $x = 2y$ and $y = 3z$

$\Rightarrow x = 6z$

$\Rightarrow \frac{x}{z} = 6$

$\therefore \frac{\text{Probability of occurence of } B_1}{\text{Probability of occurence of } B_3} = 6$

56. Sol. 540

Let $M=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$\Rightarrow M^T=\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

$\therefore MM^T=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

$\Rightarrow MM^T=\begin{bmatrix} a^2+b^2+c^2 & - & - \\ - & d^2+e^2+f^2 & - \\ - & - & g^2+h^2+i^2 \end{bmatrix}$

$\therefore a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+i^2=7$

Case I: Seven $(1's)$ and two $(0's)$, it can be done in

${}^9C_2=36$ ways

Case II: One (2) and three $(1's)$ and five $(0's)$, it can be done in

$\frac{9!}{5!3!}=504$ ways

$\therefore$ Total ways $=540$

57. Sol. 75

$\vec{c}=\lambda\left(\vec{b}\times\left(\vec{a}\times\vec{b}\right)\right)$

$=\lambda\left(\left(\vec{b}\cdot\vec{b}\right)\vec{a}-\left(\vec{b}\cdot\vec{a}\right)\vec{b}\right)$

$=\lambda\left(5\left(-\hat{i}+\hat{j}+\hat{k}\right)+2\hat{i}+\hat{k}\right)$

$= \lambda\left(-3\hat{i} + 5\hat{j} + 6\hat{k}\right)$

$\vec{c} \cdot \vec{a} = 7$

$\Rightarrow 3\lambda + 5\lambda + 6\lambda = 7$

$\Rightarrow \lambda = \frac{1}{2}$

$\therefore 2\left|\left(\frac{-3}{2} - 1 + 2\right)\hat{i} + \left(\frac{5}{2} + 1\right)\hat{j} + (3 + 1 + 1)\hat{k}\right|^2$

$= 2\left(\frac{1}{4} + \frac{49}{4} + 25\right)$

$= 25 + 50$

$= 75$

58. Sol. 10

Let $z = x + iy$

$z + \alpha|z - 1| + 2i = 0$

$x + iy + \alpha\sqrt{(x-1)^2 + y^2} + 2i = 0 + 0i$

$\therefore y + 2 = 0$ and $x + \alpha\sqrt{(x-1)^2 + y^2} = 0$

$\Rightarrow y = -2$ and $x^2 = \alpha^2\left(x^2 - 2x + 1 + 4\right)$

$\Rightarrow \alpha^2 = \frac{x^2}{x^2 - 2x + 5}$

$\alpha^2 \in \left[0, \frac{5}{4}\right]$

$\therefore \alpha^2 \in \left[0, \frac{5}{4}\right]$

$\therefore \alpha \in \left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$

Then $4\left[(\alpha_{max})^2 + (\alpha_{min})^2\right] = 4\left(\frac{5}{4} + \frac{5}{4}\right) = 10$

59. Sol. 5

3 digit number of the form $9k+2$ are $\{101,110,\ldots,992\}$

$\Rightarrow$ Sum equal to $S_1 = \frac{100}{2}(1093) = 54650$

Similarly sum of 3 digit number of the form $9k+5$, $S_2 = \frac{100}{2}(1099) = 54950$

$$S_1 + S_2 = \frac{100}{2}(1093) + \frac{100}{2}(1099) = 100 \times (1096) = 400 \times 274$$

$\Rightarrow \ell = 5$

60. Sol. 3

$$\int_{-a}^{0}(-2x+2)\,dx + \int_{0}^{2}(x+2-x)\,dx + \int_{2}^{a}(2x-2)\,dx = 22$$

$$x^2 - 2x\Big|_0^{-a} + 2x\Big|_0^2 + x^2 - 2x\Big|_2^a = 22$$

$$a^2 + 2a + 4 + a^2 - 2a - (4-4) = 22$$

$$2a^2 = 18$$

$\Rightarrow a = 3$

$$\therefore \int_{3}^{-3}(x+[x])\,dx = -\int_{-3}^{3}(x+[x])\,dx$$

$$= -(-3-2-1+1+2)$$

$$= 3$$